Chapter 1

MATHEMATICAL ANALYSIS AND CONVEXITY WITH APPLICATIONS TO ECONOMICS

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The following is intended as a guide through the basic mathematical concepts commonly used in economic theory. We have not aimed at either completeness of coverage or generality. Rather, the goal has been to provide statements of the most basic propositions in the areas of mathematics usually referred to as point-set topology and convex analysis. The reader is referred to Rudin (1964, ch. 1) and Simmons (1963, ch. 1) for notions of ordering, the real number system, inf and sup, and other concepts from elementary set theory, as these are not presented here. Occasionally we shall delve somewhat more deeply into specialized material where a result of special usefulness in economic theory is not readily available in the literature or where the proof we provide conveys insight of relevance to the economic contexts in which the result is often used. In the text of each section, propositions are proved only if they are of this latter nature. Many proofs are gathered at the ends of the sections. Still others are omitted if they are completely straightforward or if they are easily available in the literature. Suggestions for references for each section are given at the end of the main text of this Chapter.

1. Functions

Given two sets A and B, f is said to be a *function* or *mapping* from A into B if for each $x\in A$ there exists a unique $y\in B$ such that $y=f(x)$. The set A is called the *domain* of f and the subset of B consisting of points y such that $y=f(x)$ for some $x\in A$ is called the *range* of f. We write $f\colon A\to B$ to mean f is a function from A to B.

The *inverse image of a point* $y\in B$ is $\{x\,|\,x\in A,\ y=f(x)\}$; the *inverse image of a set* B', $B'\subseteq B$ is $f^{-1}(B')=\{x\,|\,x\in A,\ f(x)\in B'\}$. If for every $y\in B$ the inverse image of y is at most a single point, f is said to be a *one-to-one* mapping. If the range of f is identical to B, f is said to be an *onto* mapping.

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Definition

If A and B are two sets, their *product*, $A\times B$ is the set $\{(x, y)|x\in A, y\in B\}$.

If $A=B=\boldsymbol{R}$, the set of real numbers, then their product, $\boldsymbol{R}\times\boldsymbol{R}$, denoted $\boldsymbol{R}^2$, is the set of all ordered pairs of real numbers.

If $A=B=S^1$, the circumference of a circle of unit radius, then their product is the *torus*. A point in the torus can be described by two numbers, θ_1 and θ_2, indicating the angles around the principal axis and around the cross-section of the torus that determine its location.

When the product of many sets is needed, it is sometimes convenient to enumerate them by an index i in a set I, which may be finite or infinite. If a class of sets $\{A_i\}$ is indexed by I, then the product of this class $\Pi_{i\in I}A_i$ is the set of all functions, a, on I such that $a(i)\in A_i$.

If A is a set in a space X, then we denote by A^c, the *complement of* A, $A^c=\{x\in X|x\notin A\}$. If A and B are in X we denote $A\backslash B=\{x\in X|x\in A, x\notin B\}$.

Definition

A *projection* from $\Pi_{i\in I}A_i$ into A_i is the function mapping a into $a(i)$.

Definition

The *graph* of a function f: $A\to B$ is the set $\{(x, y)|y=f(x), x\in A\}$.

Definition

If f: $A^1\to A^2$ and g: $A^2\to A^3$, the function $g\circ f$: $A^1\to A^3$ is defined by $g\circ f(x)=g(f(x))$ for all $x\in A^1$. It is called the *composition* of the functions f and g.

2. Metric spaces

We now turn to the study of metric spaces. Here, the properties of the real numbers will be used extensively, and without proof. For a fundamental treatment of the structure of the space $\boldsymbol{R}$, the interested reader should consult Rudin (1964, ch. 1).

Definition

A *metric space* is a pair, (S, d), where S is a non-empty set and d: $S\times S\to\boldsymbol{R}$ satisfies

(i) $d(x, y)\geqslant 0$, for all $x, y\in S$,
(ii) $d(x, y)=0$, if and only if $x=y$,
(iii) $d(x, y)=d(y, x)$,
(iv) $d(x, z)\leqslant d(x, y)+d(y, z)$, for all $x, y, z\in S$.

Examples

(1) $\mathbf{R}^1 = (R^1, d)$, R^1 is the real line,
$d = |x-y|$.

To see that this is a metric space we have to verify each of the conditions. The first three are obvious from the definition of absolute value, $|\cdot|$. To check (iv), observe that there are essentially two cases:

(1) $x \geqslant y \geqslant z$ and (2) $x \geqslant z \geqslant y$,

the others involving only permutations of the letters. If (1) holds, then $|x-y| + |y-z| = |x-z|$. If (2) holds, then $|x-z| + |z-y| = |x-y|$. Since $|z-y| \geqslant 0$, we have $|x-z| \leqslant |x-y| \leqslant |x-y| + |y-z|$.

(2) $\mathbf{R}^2_m = (R^2, d_m)$, R^2 is the real plane,
$d_m(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$.

Verify (iv) as follows:

$$\begin{aligned}
d_m(x, y) &= \max\{|x_1 - z_1 + z_1 - y_1|; |x_2 - z_2 + z_2 - y_2|\} \\
&\leqslant \max\{|x_1 - z_1| + |z_1 - y_1|; |x_2 - z_2| + |z_2 - y_2|\} \\
&\leqslant \max\{|x_1 - z_1| + |z_1 - y_1|; |x_1 - z_1| + |z_2 - y_2|; \\
&\qquad |x_2 - z_2| + |z_1 - y_1|; |x_2 - z_2| + |z_2 - y_2|\} \\
&\leqslant d_m(x, z) + d_m(z, y).
\end{aligned}$$

(3) $\mathbf{R}^2 = (R^2, d_e)$, $d_e(x, y) = ((x_1 - y_1)^2 + (x_2 - y_2)^2)^{\frac{1}{2}}$.

The reader can verify that d_e defines a metric; it is called the *Euclidean metric* and corresponds to the ordinary notion of distance in the plane.

Definition

Two metric spaces $\langle S_1, d_1 \rangle$ and $\langle S_2, d_2 \rangle$ are *isometric* if there is a function f: $S_1 \to S_2$ which is one-to-one, onto and such that

$$d_2(f(x), f(y)) = d_1(x, y), \quad \text{for all} \quad x, y \in S_1.$$

Metric spaces may have sets whose elements are actually functions, or have a more complex description than just "points" like those above. Nevertheless the elements are still referred to as "points".

Examples

(1) $\mathcal{C}[0,1]$.

The set is the set of all continuous functions on

$$[0,1]=\{x \mid x\in \boldsymbol{R}, 0\leqq x\leqq 1\} \text{ into } \boldsymbol{R}.$$

The metric, $d(f,g)$, is defined as

$$d_{\mathcal{C}}=\sup_{x\in[0,1]} |f(x)-g(x)|.$$

(2) $\mathcal{C}L[0,1]$.

Same set. Metric is

$$d_{\mathcal{C}L}(f,g)=\int_0^1 |f(x)-g(x)|\,dx.$$

The proof that $\mathcal{C}$ and $\mathcal{C}L$ are metric spaces is left to the reader.

Definition

A metric space (S_1, d_1) is said to be a *subspace* of (S,d) if $S_1\subseteq S$ and $d_1(x,y)=d(x,y)$ for all $x,y\in S_1$.

Examples

(1) Let ℓ_∞ be the space of all bounded sequences in the real line, with the metric

$$d(x,y)=\sup_i |x_i-y_i|.$$

A subspace of ℓ_∞ is the metric space, c, of all convergent sequences in the real line. A subspace of c is the set of all sequences converging to zero.

(2) Consider S^2, the surface of a sphere of dimension 2, where distance is measured as the minimum length connecting the points, lying in the surface of the sphere (great circle distance). The space, S^1, which consists of a single great circle lying in the sphere is a subspace of S^2.

3. Topological structure of metric spaces

The concept of a metric is used to define a notion of distance on a space and carries with it a certain structure. The idea that points are close, measured by the metric, can be used to make precise concepts such as "boundary", "interior" and "connected", which have a geometric interpretation in ordinary language. Moreover, certain properties of sets (for example, those that contain their own boundary) are useful in developing other mathematical concepts. Therefore, these inherit their structure indirectly from the metric.

Definition

If (S,d) is a metric space and $x \in S$, the *ε-sphere about* x is given by

$$S_\varepsilon(x) = \{y \mid y \in S, d(x,y) < \varepsilon\}.$$

Definition

A subset B of a metric space (S,d) is *open* (or, *open in* S) if for every $x \in B$ there exists $\varepsilon > 0$ such that $S_\varepsilon(x) \subseteq B$.

Clearly ε-spheres are open. The empty set is open in every metric space and the metric space is open in itself. However, it is possible to have a set that is a subset of a metric space and of some proper subspace that is open in the latter, but not in the former. For instance, $(0,1]$ is open in $[0,1]$ but not in $\boldsymbol{R}^1$.

Proposition 3.1

Let (S,d) be a metric space. $B \subseteq S$ is open if and only if B is the union of ε-spheres.

Proposition 3.2

The union of open sets is open. The intersection of finitely many open sets is open.

Proof

Follows directly from the definitions.

One can discuss open sets in a more general setting than that of metric spaces by introducing the notion of a topological space. This allows us to isolate those aspects of the structure of the space which are independent of the metric. The rest of this section is devoted to discussing the implications of the metric for the topological structure of the space.

Definition

A *topological space* is a pair $(S, \mathfrak{T})$ where S is a set and $\mathfrak{T}$ is a collection of subsets of S satisfying:

(i) $\phi, S \in \mathfrak{T}$,
(ii) $B_\alpha \in \mathfrak{T}$, for all $\alpha \in I$, implies $\cup_{\alpha \in I} B_\alpha \in \mathfrak{T}$,
(iii) $B_1, \ldots, B_n \in \mathfrak{T}$ implies $\cap_{\alpha=1}^n B_\alpha \in \mathfrak{T}$.

In this definition the *open sets* are the elements of the collection $\mathfrak{T}$. They are primitives of the system in that they need not be derived from any other concept. The collection $\mathfrak{T}$ is said to be a *topology* on S.

Proposition 3.3

In any metric space, the family of all open sets is a topology for the space.

Proof

Follows from previous proposition.

It is to be noted, however, that not all topological spaces can be derived from an underlying metric on the space. For example, if

$$S=\{a,b\} \quad \text{and} \quad \mathfrak{T}=\{S,\{a\},\phi\},$$

then $\mathfrak{T}$ is a topology on S, but $\mathfrak{T}$ is not derived from any metric. This is because if d were a metric, then $d(a,b)>0$ and hence $\{a\}$ and $\{b\}$ would have to be open, being ε-spheres centered at a and b, respectively, for $\varepsilon<d(a,b)$.

Given a set S, the metric d is said to *generate* the topology $\mathfrak{T}$ if $\mathfrak{T}$ is the class of open sets defined by d.

Definition

Two metrics d_1 and d_2 on S are said to be *topologically equivalent* if they generate the same topology $\mathfrak{T}$.

Proposition 3.4

If d_1 and d_2 are topologically equivalent then for each point $x\in S$, and positive numbers ε and δ, there exist positive numbers $\bar{\varepsilon}$ and $\bar{\delta}$ such that

$$S_{\bar{\varepsilon}}^{d_1}(x)\subseteq S_{\varepsilon}^{d_2}(x) \quad \text{and} \quad S_{\bar{\delta}}^{d_2}(x)\subseteq S_{\delta}^{d_1}(x),$$

where $S_{\varepsilon}^{d_i}$ are the ε-spheres using the metric d_i.

Definition

A point x in a subset, B, of a metric space (S,d) is said to be an *interior point* of B if there exists $\varepsilon>0$ such that $S_{\varepsilon}(x)\subseteq B$. The set of all interior points of B is called the *interior* of B (denoted $\mathring{B}$ or int B).

Proposition 3.5

(a) $A\subseteq B$, A open implies $A\subseteq\mathring{B}$.
(b) B is open if and only if $B=\mathring{B}$
(c) Let $\{B_\alpha\}$ be the collection of all open sets with $B_\alpha\subseteq B$.
Then $\mathring{B}=\cup_\alpha B_\alpha$.
(d) $\mathring{A}\cap\mathring{B}=(A\mathring{\cap}B)$.
(e) $\mathring{A}\cup\mathring{B}\subseteq A\mathring{\cup}B)$.

Proof

Follows directly from the definitions.

To see that the set inclusion in part (e) may be strict, consider

$$A=[0,1], \qquad B=[1,2], \qquad S=\boldsymbol{R}^1.$$

Then,

$$\mathring{A}=(0,1), \qquad \mathring{B}=(1,2),$$

$$(A\mathring{\cup}B)=(0,2) \quad \text{but} \quad 1\notin\mathring{A}\cup\mathring{B}.$$

Definition

Let B be a subset of a metric space. A point x is said to be a *limit point* of B if for any $\varepsilon>0$, $(S_\varepsilon(x)\backslash\{x\})\cap B\neq\phi$.

The set of all limit points of B is written $\ell p(B)$. The *closure* of B is $B\cup\ell p(B)$, and is written $\overline{B}$.

A set B is said to be *closed* if $B=\overline{B}$.

Whether or not a set is closed may depend on the metric space it is considered to be a subset of. For example, $B=[\frac{1}{2},1)$ is closed in $S=[0,1]$ but not in $S=[0,1]$ or $S=\boldsymbol{R}^1$. Usually the space is understood, but it is sometimes important to be specific. In these cases one says "B is closed in S".

Proposition 3.6

(a) $\overline{\overline{B}}=\overline{B}$.
(b) B closed if and only if B^c open.
(c) Finite unions and arbitrary intersections of closed sets are closed.

Proof

Follows directly from the definitions.

Definition

A point $x\in S$ is said to be a *boundary point* of a subset B if $x\in\overline{B}$ and $x\in\overline{B^c}$. The set of all boundary points is denoted bdy$B$.

Examples

(a) Consider $S=\mathcal{C}[0,1]$ and let $B=\{f\,|\,|f(x)|\leqslant 1\}$. Then bdy$B=\{f\,|\,f(x)=+1$ or $f(x)=-1$ for some x and $|f(x)|\leqslant 1$ for all $x\}$.

Proof

Since $B\subseteq\overline{B}$ and any f in the indicated set is in B, it is also in $\overline{B}$. We must show that f is a limit point of B^c. If $f(x)=+1$ for some x consider $g_\varepsilon(x)=f(x)+\varepsilon$. For

any $\varepsilon > 0$, $g_\varepsilon(x) \in B^c$. Given $\delta > 0$, any $\varepsilon < \delta$ has the property that $d(g,$
$\sup_x |g_\varepsilon(x) - f(x)| = \varepsilon$. Therefore $g_\varepsilon \in S_\varepsilon(f)$, and hence $g_\varepsilon \in S_\varepsilon(f) \setminus \{f\} \cap B^c$. T
$f \in \ell p(B^c)$.

(b) Let $S = \boldsymbol{R}^1$ and $B = \boldsymbol{Q}$, the set of all rational numbers. $\mathrm{Bdy}\boldsymbol{Q} = \boldsymbol{R}^1$ since ε-sphere in $\boldsymbol{R}^1$ contains both rationals and irrationals. This is an exar where the boundary is strictly larger than the set itself.

Proof of Proposition 3.1

If B is open in S, then for each $x \in B$ there exists ε_x such that $S_{\varepsilon_x}(x) \subseteq B$. T

$$\bigcup_{x \in B} S_{\varepsilon_x}(x) \subseteq B,$$

and clearly,

$$\bigcup_{x \in B} S_{\varepsilon_x}(x) \supseteq B.$$

Therefore B is the union of ε-spheres. Conversely, let

$$B = \bigcup_{\alpha} S_{\varepsilon_\alpha}(x_\alpha) \quad \text{for} \quad \alpha \in I,$$

an indexing set. If $I = \phi$, then $B = \phi$ and is therefore open. If $I \neq \phi$, then e $x \in B$ is in $S_{\varepsilon_{\alpha_x}}(x_{\alpha_x})$ for some $\alpha_x \in I$. Let

$$r_x = \varepsilon_{\alpha_x} - d(x, x_{\alpha_x}).$$

Because d is a metric and hence satisfies the triangle inequality,

$$S_{r_x}(x) \subseteq B.$$

Therefore the existence of r_x as defined for each $x \in B$ suffices to show that open. ■

Proof of Proposition 3.4

Let $\mathfrak{T}$ be the topology generated by both d_1 and d_2. In particular, $S_\varepsilon^{d_2}(x) \in \mathfrak{T}$ all ε and x.

But, $S_\varepsilon^{d_2}(x)$ is open in the topology generated by d_1. Thus there is an o sphere in the d_1 metric centered on x and of sufficiently small radius $\bar{\varepsilon}$ such t $S_{\bar{\varepsilon}}^{d_1}(x) \subseteq S_\varepsilon^{d_2}(x)$.

Reversing the roles of d_1 and d_2, the proof of the proposition is complete.

4. Sequences, complete spaces, and separable spaces

Let Z^+ be the non-negative integers, in increasing order.

Definition

A *sequence* in (S,d) is a mapping

$$s: Z^+ \to S.$$

Definition

A sequence s has *limit* $x \in S$, if for any $\varepsilon > 0$ there exists $N \in Z^+$ such that $n \geqslant N$ implies $s(n) \in S_\varepsilon(x)$. If s has a limit x, it is said that *s converges to x*.

Definition

A *subsequence* of the sequence s is a sequence of the form

$$s \circ g,$$

where g is a strictly monotonic increasing function from Z^+ to itself. Such a sequence is written $\{s_{g_k}\}$, meaning that

$$s \circ g(k) = s(g_k).$$

For example one can take a subsequence consisting of every other member of the sequence s, in which case $g_k = 2k$.

Sequences and subsequences are more often written with the index as a subscript than as the argument of a function — i.e., (s_k).

Proposition 4.1

Let s be a sequence and $x \in \ell p\{s_1, \ldots\}$. There exists a subsequence of s, s', converging to x.

Proof

Take a sequence of positive numbers $\varepsilon_k \to 0$. There exists an integer g_1 such that $s_{g_1} \in S_{\varepsilon_1}(x)$. Moreover there exists $g_2 > g_1$ such that $s_{g_2} \in S_{\varepsilon_2}(x)$, for if not, the sequence $s_{g_1}, s_{g_1+1}, \ldots$ would not have x as a limit point, and hence neither would s. Proceeding in this way we define the required increasing sequence $g = (g_k)$ such that $s \circ g$ converges to x.

Definition

A sequence s is said to be a *Cauchy sequence* if for any $\varepsilon > 0$ there exists an integer N such that $n, m \geqslant N$ implies $d(s_n, s_m) < \varepsilon$.

Definition

A metric space is *complete* if every Cauchy sequence in it converges to some point in the space.

One of the basic properties of $\boldsymbol{R}^n$ is that it is a complete metric space. feature is "built into" the structure of the set $\boldsymbol{R}^n$ in a certain sense, rather being a "derived" property. The set $\boldsymbol{R}$ is constructed from the natural num by first forming the rational numbers and then taking their "completi Different axiomatizations perform this procedure in slightly different ways, the result in each case is a complete space. Then $\boldsymbol{R}^n$ is constructed as the n- product of $\boldsymbol{R}$, and it is easy to see that it is complete.

Proposition 4.2

Let $(A,d)\subseteq(S,d)$ and let (S,d) be complete. Then (A,d) is complete if only if A is closed as a subset S.

Let $\mathfrak{B}[0,1]$ be the space of all bounded real-valued functions on $[0,1]$.

Proposition 4.3

$\mathcal{C}[0,1]$ is a closed subspace of $\mathfrak{B}[0,1]$.

Corollary

$\mathcal{C}[0,1]$ is complete.

Examples

(1) $\mathcal{C}L[0,1]$ is *not* complete. (Take $f_n(x)=1$ for $x>2^{-n}$, $f_n(x)=x/2^{-n}$ $0\leqq x\leqq 2^{-n}$.)
(2) c_{00}, the space of sequences with only a finite number of non-zero ter not complete. (Take $s_n=(1,1/2,1/3,\dots,1/n,0,0\dots)$.)
(3) Let c_0 be the space of sequences converging to zero; c_0 is complete, $c_{00}\subseteq c_0$; therefore c_{00} is not closed.

Definition

Let B and A be subsets of (S,d). B is said to be *dense in A* if $A\subseteq\overline{B}$. If we $A=S$, then B is said to be *dense* (or everywhere dense, or dense in S). B is to be *nowhere dense* if $\operatorname{int}\overline{B}=\phi$.

Definition

A set is said to be *countable* if it can be placed into a one-to-one relatior with the natural numbers.

Examples

(1) The space of rational numbers, $\boldsymbol{Q}$, is countable. This can be seei arranging the rationals in a two-dimensional lattice according to

numerator and denominator. Then, quotients that are redundant are deleted, and the rest can be enumerated by reading them off the diagonals of the array

$$\begin{array}{cccccc}
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \cdots \\
\frac{2}{1} & \frac{2}{2} & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} & \\
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} & \frac{3}{5} & \\
\frac{4}{1} & & & & &
\end{array}$$

(2) The real numbers are *not* countable. The usual proof of this proposition involves writing each real number between 0 and 1 in its binary expansion

$$0.a_1a_2a_3\ldots = \sum_{i=1}^{\infty} a_i 2^{-i},$$

where a_i is either 0 or 1 for each i. One must note that there can be at most two distinct binary expansions representing the same real number: when $a_i=0$ for $i>k$ and $a_k=1$, the same number is given by $a'_k=0$, $a'_i=1$ for $i>k$, and $a'_i=a_i$ for $i<k$. If the real numbers are countable, we can make a list of all alternative expressions for all real numbers in an array

$$\begin{aligned}
&0.a_1^1a_2^1\ldots,\\
&0.a_1^2a_2^2\ldots,\\
&0.a_1^3a_2^3\ldots.
\end{aligned}$$

Now construct a new binary decimal $0.a_1a_2\ldots$ by setting

$$\begin{aligned}
a_i&=0 \quad \text{if} \quad a_i^i=1,\\
a_i&=1 \quad \text{if} \quad a_i^i=0.
\end{aligned}$$

Clearly the real number represented by $0.a_1a_2\ldots$ is different from any of the members in the list. Hence the real numbers cannot be counted in this way.

Definition

The space (S, d) is said to be *separable* if it contains a countable dense subset. Separable spaces are very useful, particularly when approximation results are desired. For this purpose one takes a subset of the countable dense set that "comes close to filling the space"...

Examples

(1) A is dense in A for any set $A \subseteq S$.
(2) $\boldsymbol{Q}$ is dense in $\boldsymbol{R}$.
(3) $\boldsymbol{Q}$ is dense in $(0,1]$ as a subset of $\boldsymbol{R}$.
(4) $\boldsymbol{Z}^+$ is nowhere dense in $\boldsymbol{R}$.
(5) Any space with countably many points is separable.

Theorem

Let (S,d) be a metric space. Then there is a complete metric space, $(S,d)^*$, which is unique up to isometry, such that (S,d) is isometric to a dense subset of $(S,d)^*$.

For a proof, see Kolmogorov and Fomin (1970).

Example

If (S,d) is $\boldsymbol{Q}$, then $(S,d)^*$ can be taken to be $\boldsymbol{R}$.

Examples of separable spaces

(1) $\boldsymbol{R}^1$
(2) $\boldsymbol{Z}, \boldsymbol{Q}$
(3) $\boldsymbol{R}^n$
(4) $\mathcal{C}[0,1]$ (Take polynomial functions with rational coefficients, or linear piecewise functions with graphs having kinks only at points in $\boldsymbol{Q}\times\boldsymbol{Q}$.)
(5) $\mathcal{B}[0,1]$ and ℓ_∞ are *not* separable.

We say a sequence of sets $\{B_n\}_{n=1,\ldots}$ is *decreasing* if $B_n \supseteq B_{n+1}$ for all n. Also, define the *diameter* of a set $d(A)=\sup\{d(x,y)|x,y\in A\}$.

Theorem 4.4 (Cantor Intersection Theorem)

Let $\{B_n\}_{n=1,\ldots}$ be a decreasing sequence of non-empty, closed subsets of a complete metric space (S,d) such that $d(B_n)$ converges to zero. Then $B=\cap_{n=1}^{\infty} B_n$ contains exactly one point.

Proof of Proposition 4.2

Necessity. Assume (A,d) is complete. We will show that $x\in \ell p(A)$ implies $x\in A$. By virtue of Proposition 4.1, we construct a sequence $\{x_n\}$ in A such that x_n converges to $x\in S$ and such that $d(x,x_n)<1/n$. Such a sequence is Cauchy. Since A is complete, $x\in A$.

Sufficiency. Assume A is closed and (x_n) is a Cauchy sequence in A. Since the metric in A and S is the same, (x_n) is also Cauchy in S; and since S is complete it converges, say to $x \in S$. Suppose $x \notin A$, now $x \in \ell p(A)$, and A is closed. Therefore $x \in A$, and x_n converges to a point in A, proving that A is complete. ■

Proof of Proposition 4.3

Take $f \in \ell p\mathcal{C}[0,1]$. We will show $f \in \mathcal{C}[0,1]$. Let (f_n) be a sequence in $\mathcal{C}[0,1]$ converging to f. Since $\mathcal{B}[0,1]$ is complete, $f \in \mathcal{B}[0,1]$. We must show that f is continuous—that is, given $x \in [0,1]$ and $\varepsilon > 0$ there is $\delta > 0$ such that for $r \in S_\delta(x)$, $|f(x) - f(r)| < \varepsilon$. By the triangle inequality,

$$|f(x)-f(r)| \leqslant |f(x)-f_n(x)| + |f_n(x)-f_n(r)| + |f_n(r)-f(r)|.$$

Take $d(f, f_n) < \varepsilon/3$ for $n \geqslant N$. Thus the first and third terms are made smaller than $\varepsilon/3$. Since $f_n \in \mathcal{C}[0,1]$, there exists $\delta > 0$ such that

$$|f_n(x) - f_n(r)| < \varepsilon/3,$$

when $|x-r| < \delta$. Taking $|x-r| < \delta$ and $n \geqslant N$, $|f(x)-f(r)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$. Therefore $f \in \mathcal{C}[0,1]$, and $\mathcal{C}[0,1]$ is therefore closed. ■

Proof of Theorem 4.4

For each n, select $x_n \in B_n$. $\{x_n\}$ is a Cauchy sequence since $d(B_n)$ converges to zero. Since (S, d) is complete, x_n converges to x. To show that $x \in B_n$ for all n: Suppose $x \notin B_k$ for some k. Then $x \in B_k^c$ and B_k^c is open in S as B_k is closed. But since $\{B_n\}$ is a decreasing sequence, $B_k^c \cap B_n = \phi$ for $n \geqslant k$.

This contradicts $\lim x_n = x$ since B_k^c is an open set around x which is disjoint from $x_n (\in B_n)$ for n sufficiently large. Thus $\cap B_n \neq \phi$. There cannot be two points in it, as $d(B_n)$ can be made arbitrarily small. ■

5. Continuity

One of the most important topological concepts is that of continuity of functions. Intuitively, we want to express the idea that a function varies only slightly when its argument varies slightly. But the idea of small variation is connected to the metric on the space. For large, complex spaces, this may be a subtle issue. It turns out that continuity is in a sense equivalent to the definition of topology. The basic structure of spaces is preserved under continuous deformation. Continuous functions also have other features of particular importance to economics — for example, they are needed to assert the existence of maxima and of fixed points (see below). Therefore, the concept of continuity is one of the central mathematical ideas to be studied.

Definition

A mapping f from a metric space (S, d) into a metric space (T, ρ) is *continuous at* $x \in S$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $\rho(f(x), f(y)) < \varepsilon$.

If f is continuous at x for all $x \in S$, then f is said to be *continuous*.

Theorem 5.1

Let f: $(S, d) \to (T, \rho)$. The following are equivalent:

(i) f is continuous.
(ii) For all $x \in S$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $f(S_\delta(x)) \subseteq S_\varepsilon(f(x))$.
(iii) For all open sets B in (T, ρ), $f^{-1}(B)$ is open in (S, d).
(iv) If $\{x_n\}$ converges to x in S then $f(x_n)$ converges to $f(x)$ in T.

Proposition 5.2

Let f: $(R, \gamma) \to (S, d)$ and g: $(S, d) \to (T, \rho)$ be continuous functions. Then $g \circ f$: $(R, \gamma) \to (T, \rho)$ is continuous.

Proof

Immediate on applying (iii) in the previous proposition.

Definition

Let f: $(S, d) \to (T, \rho)$. f is said to be *uniformly continuous* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(x, y) < \delta$, then $\rho(f(x), f(y)) < \varepsilon$. (Here δ depends on ε but not on x.)

Proposition 5.3

Let f be uniformly continuous, then $\{x_n\}$ Cauchy implies $\{f(x_n)\}$ Cauchy.

Proof

Immediate from definition.

Definition

A map f: $(S, d) \to (T, \rho)$ is a *homeomorphism* if it is one-to-one, onto, continuous and f^{-1} is continuous.

If there exists a homeomorphism between (S, d) and (T, ρ) they are said to be *homeomorphic*. Any property preserved under homeomorphisms is said to be a *topological property*.

In the special case where (S, d) is homeomorphic to (S, ρ), the metrics d and ρ are topologically equivalent. This can be seen from the definition in Section 3 and Theorem 5.1.

Proof of Theorem 5.1

(i) implies (ii). Follows directly upon noting that $S_\delta(x)=\{x'|d(x,x')<\delta\}$ and $S_\varepsilon(f(x))=\{y=f(x')|\rho(f(x'),f(x))<\varepsilon\}$.

(ii) implies (iii). Take B open in (T,ρ), and $a\in f^{-1}(B)$. Since B is open, there exists $\varepsilon_a>0$ such that $S_{\varepsilon_a}(f(a))\subseteq B$. By virtue of (ii) there exists $\delta_a>0$ such that $f(S_{\delta_a}(a))\subseteq S_{\varepsilon_a}(f(a))\subseteq B$. Applying f^{-1} to this set inclusion, $S_{\delta_a}(a)\subseteq f^{-1}(B)$. Thus $f^{-1}(B)$ contains an open sphere about each of its points and is therefore an open set.

(iii) implies (iv). Take $\{x_n\}$ converging to $x\in(S,d)$. Let $B\subseteq(T,\rho)$ be an arbitrarily small open set containing $f(x)$. By virtue of (iii) $f^{-1}(B)$ is open and contains x. Thus there exists N such that $n\geqslant N$ implies $x_n\in f^{-1}(B)$.

Let $\varepsilon>0$ be any number such that $S_\varepsilon(x)\subset f^{-1}(B)$. Since $x_n\to x$, we know that there exists N' such that $x_n\in S_\varepsilon(x)$ for all $n\geqslant N'$. But then $x_n\in f^{-1}(B)$ so $f(x_n)\in B$ for all $n\geqslant N'$. Since B was an arbitrarily small open set containing $f(x)$, $\{f(x_n)\}$ converges to $f(x)$.

(iv) implies (i). Assume that f is not continuous at $x\in S$. Then for some $\varepsilon>0$ and each $\delta>0$ there exists $y\in S$ with $d(x,y)<\delta$ but $\rho(f(x),f(y))\geqslant\varepsilon$. Take $\{x_n\}$ such that $d(x,x_n)<1/n$ and $\rho(f(x),f(x_n))\geqslant\varepsilon$. Then x_n converges to x but $\{f(x_n)\}$ does not converge to $f(x)$, violating (iv).

6. Compactness

The notion of compactness is a basic topological tool. Indeed, it is possible to obtain all of the results in topology by taking the compact sets as the basic primitive concept instead of the open sets. The usefulness of compactness derives from the special properties possessed by continuous functions defined on compact sets, by the maximizers of real-valued functions on compact sets, and by the usefulness of alternative characterizations of compact sets and the analytical ease with which they can be checked. In addition to compact sets in Euclidean spaces, we pay special attention to function spaces and to the characteristics of compact sets in that domain.

Definition

A collection $\{A_i\}_{i\in I}$ is a *cover* of a set B if $B\subseteq\bigcup_i A_i$. An *open cover* is a cover consisting only of open sets. A cover $\{A_i\}_{i\in I}$ of B is called a *subcover* of $\{A_j\}_{j\in J}$ if for each $i\in I$ there is a $j\in J$ such that $A_i=A_j$.

Definition

A set B in (S, d) is *compact* if every open cover of B contains a finite subcover. A *compact metric space* (S, d) is one in which S is a compact subset of itself.

Proposition 6.1

Let (S, d) be a compact metric space, then $A \subseteq S$ is compact if and only if A is closed.

Proposition 6.2

A compact subset in any metric space is bounded.

Definition

Let (S, d) be a metric space. Given $\varepsilon > 0$ an *ε-net* is a finite subset F of S such that $S = \cup_{x \in F} S_\varepsilon(x)$. The space (S, d) is said to be *totally bounded* if every $\varepsilon > 0$ has an ε-net.

Proposition 6.3

A compact subset of a metric space is totally bounded.

Proof

Take $\{S_\varepsilon(x)\}_{x \in A}$, A compact. Then the centers of the finite subcover suffice for an ε-net.

Definition

The metric space (S, d) is *sequentially compact* if every sequence in (S, d) contains a convergent subsequence.

Proposition 6.4

A sequentially compact metric space (S, d) is totally bounded.

Definition

A *Lebesgue number for the open cover* $\{U_i\}$ of the space (S, d) is a real number $L > 0$ such that if $A \subseteq S$ and $d(A) < L$ then $A \subseteq U_i$ for some i.

Example

If $s = [0, 1]$, $d = |x - y|$ is the usual metric, and $\{U_i\} = \{[0, \frac{3}{4}), (\frac{1}{4}, 1]\}$ is an open cover of S, then any $L < \frac{1}{2}$ is a Lebesgue number for $\{U_i\}$. Consider a set A containing points $x \leqslant \frac{1}{4}$ and $y \geqslant \frac{3}{4}$, so that $A \not\subset U_i \in \{U_i\}$, but then $d(A) \geqslant \frac{1}{2}$. Conversely, if $d(A) < \frac{1}{2}$ then it must be true that either $x < \frac{3}{4}$ or $x > \frac{1}{4}$ for all $x \in A$, and therefore either $A \subset [0, \frac{3}{4})$ or $A \subset (\frac{1}{4}, 1]$.

Notice that if the cover is not open, there may be no Lebesgue number. Consider as an example $\{U_i\}=\{[0,\frac{1}{2}],[\frac{1}{2},1]\}$. For any $\varepsilon>0$, $[\frac{1}{2}-\varepsilon,\frac{1}{2}+\varepsilon]$ is not contained in either $U_i\in\{U_i\}$. Strict positivity of the Lebesgue number is the important feature.

Proposition 6.5

Every open cover of a sequentially compact metric space has a Lebesgue number.

Definition

The metric space (S,d) has the *Bolzano–Weierstrass property* if every infinite subset of S has a limit point. (This point does not necessarily lie in the subset.)

Theorem 6.6

Let (S,d) be a metric space. The following are equivalent:

(i) (S,d) is compact.
(ii) (S,d) has the Bolzano–Weierstrass property.
(iii) (S,d) is sequentially compact.
(iv) (S,d) is totally bounded and complete.

Proof

(i) implies (ii). Let A be a subset of S containing an infinite number of points but no limit point. Take $x\in A$.

There exists $\varepsilon_x>0$ such that

$$\left(S_{\varepsilon_x}(x)\backslash\{x\}\right)\cap A=\phi.$$

Doing this for all $x\in A$, we have that $\{A^c, S_{\varepsilon_x}(x)\}_{x\in A}$ is an open cover of S with no finite subcover. Therefore A has limit points and the Bolzano–Weierstrass property is validated.

(ii) implies (iii). Assume (ii) holds and let $\{x_n\}$ be a sequence in S. If the set $\{x_1,\ldots\}$ has only finitely many distinct points, then there exists a constant, convergent, subsequence. If $\{x_1,\ldots\}$ is infinite, then it has a limit point by virtue of (ii). Thus we can find a subsequence with this point as its limit, and (S,d) is sequentially compact.

(iii) implies (iv). By Proposition 6.4, (S,d) is totally bounded. We next prove completeness. Any Cauchy sequence has a subsequence that converges to some point b, by virtue of (iii). For any $\varepsilon>0$, there exists N such that $d(x_n,x_m)<\varepsilon/2$ for $n,m\geqslant N$. Also, $d(b,x_p)<\varepsilon/2$ for some $p>N$, by definition of b. Hence, $n>N$ implies $d(b,x_n)\leqslant d(b,x_p)+d(x_p,x_n)<\varepsilon$, as was to be shown.

We next show that condition (iii) implies (i). After that, we shall prove that (iv) implies (iii), thereby completing the proof of the theorem.

(iii) implies (i). Let (S, d) be sequentially compact and let $\{U_i\}_{i\in I}$ be an open cover of S. By Proposition 6.5, $\{U_i\}$ has a Lebesgue number $L>0$. Then there exists for S an ε-net, $\{x_1,\dots\}$, with $\varepsilon=L/3$, so that $\{S_\varepsilon(x_1),\dots,S_\varepsilon(x_N)\}$ covers S. Observe that $d(S_\varepsilon(x_k))\leqslant 2\varepsilon=\frac{2}{3}L<L$. Therefore for each $k=1,\dots,N$, there is $i_k\in I$ such that $S_\varepsilon(x_k)\subseteq U_{i_k}$. Hence $\{U_{i_k}\}_{k=1,\dots,N}$ forms a finite open cover of S, and S is thus compact.

(iv) implies (iii). We shall show that every sequence has a Cauchy subsequence. Since S is complete, this will be sufficient. Consider any sequence of points in S, $\{x_i^1\}_{i=1}^\infty$. Since S is totally bounded, there is a finite set of open spheres, each with radius $\frac{1}{2}$, which cover S. There is therefore a subsequence of the original sequence, say $\{x_i^2\}_{i=1}^\infty$, which lies entirely in one of the open spheres of radius $\frac{1}{2}$. Similarly, we can argue that there is a subsequence of $\{x_i^2\}$ which lies entirely in a sphere of radius $\frac{1}{3}$, called $\{x_i^3\}$. This procedure may be continued indefinitely, with $\{x_i^n\}$ contained in a sphere of radius $1/n$. Now, consider the diagonal subsequence of $\{x_i^1\}_{i=1}^\infty:\{x_1^1,x_2^2,x_3^3,\dots\}$. This is a Cauchy subsequence of the original sequence; for if $n>m$, then $d(x_m^m,x_n^n)<1/m$. ■

Proposition 6.7

The continuous image of a compact set is compact.

The above proposition, when combined with Theorem 6.6, has a very useful corollary in many economic applications.

Corollary

Let (S, d) be a compact metric space and let f: $S\to \boldsymbol{R}$ be a continuous function. Then there exists $x\in S$ such that $f(x)\geqq f(y)$ for all $y\in S$.

In many economic contexts the relevant objects of choice are functions rather than numbers or vectors. Examples might be the rate of investment over time, tax rates as a function of income or probabilities (in a mixed strategy) as a function of an observable, continuously divisible, state of the system. It may be important to prove that the feasible set of such actions is compact. (See Sections 10 and 11 below.) We therefore study the structure of the space of continuous functions.

Definition

Let $\mathfrak{F}$ be a collection of functions from a metric space A to a metric space B, with metric d. $\mathfrak{F}$ is said to be *equicontinuous* at $x_0\in A$ if for each $\varepsilon>0$ there is a neighborhood U of x_0 such that

$$d(f(x), f(x_0))<\varepsilon \quad \text{for any} \quad f\in\mathfrak{F} \quad \text{and any} \quad x\in U.$$

If f *is* equicontinuous at all $x_0\in A$, then $\mathfrak{F}$ is *equicontinuous*.

Lemma

Let A be a compact metric space and let B be a compact metric space with metric d. A collection of continuous functions $\mathfrak{F}$ from A to B is equicontinuous if and only if $\mathfrak{F}$ is totally bounded under the metric ρ given by

$$\rho(f,g)=\sup_{x\in A} d(f(x)-g(x)).$$

Theorem 6.8 (Ascoli's Theorem)

Let A be a compact space. A subset $\mathfrak{F}$ of the continuous functions from A into $\boldsymbol{R}^n$ is compact if and only if it is closed, bounded and equicontinuous.

Proof

Assume $\mathfrak{F}$ is compact. By Propositions 6.1 and 6.2 it is closed and bounded. By Theorem 6.6 ((i) implies (iv)) it is totally bounded. By the Lemma, it is therefore also equicontinuous.

Conversely, assume $\mathfrak{F}$ is closed, bounded and equicontinuous. We will now prove it is compact by demonstrating that it is sequentially compact and again applying Theorem 6.6 ((iii) implies (i)). Since each $f\in\mathfrak{F}$ is bounded, Proposition 4.3 and its corollary imply that $\mathfrak{F}$ is complete.

Take a sequence $\{f_i\}$ in $\mathfrak{F}$. It will suffice to show that there is a Cauchy subsequence. The compactness of A implies that it is totally bounded. We can find a countable dense subset of A (for example the centers of the spheres defining ε-nets for a sequence of ε's converging to zero). Let this dense subset be arranged in some sequence $\{x_i\}$.

Consider the sequence of points in $\boldsymbol{R}^n$ defined by $\{f_i(x_1)\}$. Since $\mathfrak{F}$ is bounded, there is a convergent subsequence. Call it $\{f_{i1}(x_1)\}$. Now consider $\{f_{i1}(x_2)\}$. It will contain a convergent subsequence $\{f_{i2}(x_2)\}$. Continue in this way and consider the subsequence composed of the "diagonal" elements $\{f_{ii}\}$ where f_{ii} is the ith function in the ith subsequence. We will show that $\{f_{ii}\}$ is Cauchy.

Fix $\varepsilon>0$. Since $\mathfrak{F}$ is equicontinuous there exists $\delta>0$ such that if $d(x,x')<\delta$ then $\|f_{ii}(x)-f_{ii}(x')\|<\varepsilon/3$, for all f_{ii} in the diagonal subsequence.

Consider the δ-spheres centered on the x_i. They are an open cover of A and a finite subcover exists since A is compact. Let I be the largest index of the x_i generating this subcover. For any $x\in A$ $\inf_{i\leq I} d(x,x_i)<\delta$. We can find an N such that if $n,m\geqslant N$ then $\|f_{mm}(x_j)-f_{nn}(x_j)\|<\varepsilon/3$, as $\{f_{ii}(x_j)\}$ converges for each x_j (being a subsequence of a converging sequence). Therefore, for $m,n\geqq N$ and any $x\in A$, we can write

$$|f_{mm}(x)-f_{nn}(x)|\leqq|f_{mm}(x)-f_{mm}(x_i)|+|f_{mm}(x_i)-f_{nn}(x_i)|+|f_{nn}(x_i)-f_{nn}(x)|,$$

and if $d(x,x_i)<\delta'$, each of the terms on the right-hand side will be bounded by $\varepsilon/3$. Thus $\{f_{ii}\}$ is Cauchy. ■

Proof of Proposition 6.1

Sufficiency. Take $x \in A^c$. We will show that A^c is open by demonstrating the existence of an open set U containing x such that $U \cap A = \phi$. For each $y \in A$ choose $\varepsilon_y = \frac{1}{2} d(x, y)$. Then $\{S_{\varepsilon_y}(y)\}_{y \in A}$ is an open cover of A. Let $(S_1, ..., S_N)$ be a finite subcover of A. Such a subcover exists because A is assumed to be compact. We know that $x \notin \bar{S}_{\varepsilon_y}(y)$ and so $x \notin \bar{S}_k$ for any $k = 1, \ldots, N$. But then, $x \notin \cup \bar{S}_k$ and since this is a finite union of closed sets it is closed. Letting $U = (\cup \bar{S}_k)^c$ we have $x \in U$ and $U \cap A = \phi$.

Necessity. Let C be closed. Let $\{U_i\}_{i \in I}$ be any open cover of C. Since C^c is open, $\{U_i\}_{i \in I} \cup \{C^c\}$ is an open cover of S. Thus there exists a finite subcover of C, and hence it is compact. ■

Proof of Proposition 6.2

Let A be compact. Then $\{S_1(x)\}_{x \in A}$, a set of spheres with radius 1, is an open cover of A. By compactness there is a finite subcover, say with N spheres of radius 1 and centers, x_i, $i = 1, \ldots, N$. Then

$$d(A) \leqslant 2 + \max_{1 < i, j < N} d(x_i, x_j),$$

which is finite. ■

Proof of Proposition 6.4

Take $\varepsilon > 0$, and $x_1 \in S$. If $S = S_\varepsilon(x_1)$ then $\{x_1\}$ is an ε-net. If not, choose $x_2 \in (S_\varepsilon(x_1))^c$. If $S = S_\varepsilon(x_1) \cup S_\varepsilon(x_2)$ then $\{x_1, x_2\}$ is an ε-net. Continuing in this way, we note that the process must terminate since otherwise $\{x_1, \ldots\}$ would be a sequence with no convergent subsequence, contrary to hypothesis. Whenever it stops, the set $\{x_1, \ldots, x_N\}$ generated up to that point is an ε-net. Thus S is totally bounded. ■

Proof of Proposition 6.5

Let $\{U_i\}_{i \in I}$ be an open cover and suppose there is no Lebesgue number. Then there exists $\{A_n\}$ a sequence of subsets of (S, d) such that $d(A_n) = \delta_n$ and δ_n converges to zero, where no A_n is contained in any U_i. Choose a_n in A_n and define the sequence $\{a_n\}$. Since (S, d) compact there is a convergent subsequence $\{a'_n\}$. Let $p = \lim_i a'_i$. As $p \in S$ we know $p \in U_i$ for some i. Since U_i open, there exists $\varepsilon > 0$ such that $S_\varepsilon(p) \subseteq U_i$. Choose K such that $i \geqslant K$ implies

$$(1) \quad d(a_{n(i)}, p) < \varepsilon/2, \qquad (2) \quad \delta_{n(i)} < \varepsilon/2.$$

Now if $x \in A_N$,

$$d(x, p) \leqslant d(x, a_{n(K)}) + d(a_{n(K)}, p) \leqslant d(A_{n(K)}) + \varepsilon/2 = \varepsilon.$$

Therefore $A_N \subseteq S_\varepsilon(p) \subseteq U_i$, which is a contradiction. Thus, every open cover has a Lebesgue number, if (S, d) is sequentially compact. ■

Proof of Corollary

Let f: $(S, d) \to (T, \rho)$ be continuous and take A compact in S. Let $\{U_i\}_{i\in I}$ be an open cover of $f(A)$. Since the sets U_i are open, $f^{-1}(U_i)$ is open by the continuity of f. Thus $\{f^{-1}(U_i)\}_{i\in I}$ is an open cover of A. A finite subcover exists, say $\{f^{-1}(U_{i_1}),\ldots, f^{-1}(U_{i_N})\}$. Take $\{ff^{-1}(U_{i_1}),\ldots, ff^{-1}(U_{i_N})\} = \{U_{i_1},\ldots, U_{i_N}\}$. This forms a finite subcover of $f(A)$, as required. ■

Proof of Proposition 6.7

Since S is compact, $f(S)$ is compact by Proposition 6.7. Let $a = \sup_{x\in S} f(x)$. For each $\varepsilon > 0$ there exists $x_\varepsilon \in S$ such that $a - f(x_\varepsilon) < \varepsilon$. Let (ε_i) be a sequence converging to zero and let (x_{ε_i}) be the sequence of points corresponding to ε_i as above. Since $f(S)$ is compact, by Theorem 6.6, $f(S)$ is sequentially compact. The sequence $(f(x_{\varepsilon_i}))$ has a subsequence, converging to $a \in f(S)$. Therefore, any element of $f^{-1}(a)$ suffices to maximize f over S. ■

Proof of Lemma

Given x_0 we first show that $\mathfrak{F}$ is equicontinuous at x_0. Let $\varepsilon > 0$ be given, and take ε_1, ε_2 positive and such that $2\varepsilon_1 + \varepsilon_2 \leqslant \varepsilon$. Cover $\mathfrak{F}$ by an ε_1-net,

$$S_{\varepsilon_1}(f_1),\ldots, S_{\varepsilon_1}(f_n).$$

Because $f_1,\ldots, f_n$ are continuous we can take an open neighborhood U of x_0 such that $x \in U$ implies

$$d(f_i(x), f_i(x_0)) < \varepsilon_2 \quad \text{for} \quad i = 1,\ldots, n.$$

Take $f \in \mathfrak{F}$. Then $f \in S_{\varepsilon_1}(f_k)$ for some k. Hence

$$\begin{aligned} d(f(x), f(x_0)) &\geqq d(f(x), f_k(x)) + d(f_k(x), f_k(x_0)) + d(f_k(x_0), f(x_0)) \\ &\geqq \quad \varepsilon_1 \quad + \quad \varepsilon_2 \quad + \quad \varepsilon_1 \\ &\geqq \quad \varepsilon. \end{aligned}$$

Therefore $\mathfrak{F}$ is equicontinuous at x_0.

Pick $\varepsilon_1 > 0$. Because $\mathfrak{F}$ is equicontinuous, for any $x_0 \in A$ we can pick a neighborhood $U(x_0)$ such that $d(f(x), f(x_0)) < \varepsilon_1$ for all $f \in \mathfrak{F}$, $x \in U(x_0)$. Let $\{U(x_0)\}_{x_0\in A}$ be the set of these neighborhoods. This set is an open cover of A. By compactness, it has a finite subcover, say $U(x_1),\ldots, U(x_m)$. Within this subcover we still have, for any $x \in U(x_i)$,

$$d(f(x), f(x_i)) < \varepsilon_1 \quad \text{for any} \quad f \in \mathfrak{F}.$$

Cover B by finitely many open sets $V_1,\ldots, V_n$ of diameter less than ε_2.

Let T be the set of all mappings τ of $\{1,\ldots,m\}$ into $\{1,\ldots,n\}$. For each τ we ask whether there is an $f\in\mathfrak{F}$ such that $f(x_i)\in V_{\tau(i)}$ for all i. If so, denote one such function by f_τ. Thus there will be a finite collection of functions $\{f_\tau\}$ for those τ in T such that this property holds within $\mathfrak{F}$.

Take $f\in\mathfrak{F}$. We will show that $f\in S_\varepsilon(f_\tau)$ for some f_τ.

Let $x\in A$ and choose i so that $x\in U_i$. Then

$$d(f(x),f(x_i)) <\varepsilon_1,$$
$$d(f(x_i),f_\tau(x_i))<\varepsilon_2,$$
$$d(f_\tau(x_i),f_\tau(x))<\varepsilon_1.$$

Since ε_1 and ε_2 are arbitrary, we can set them small enough that, as above, $d(f(x),f_\tau(x))<\varepsilon$. Because this holds for each $x\in X$,

$$\rho(f,f_\tau)=\sup\{d(f(x),f_\tau(x))\}<\varepsilon. \quad \blacksquare$$

7. Connected sets

A continuous curve in a topological space X is the image of the unit interval in $\boldsymbol{R}$ under a continuous function $f:[0,1]\to X$. A metric space X is *arc-connected* if there is a continuous curve linking any two points in X. Clearly, the set $S=(0,\frac{1}{2})\cup(\frac{1}{2},1)$ is not arc-connected in $\boldsymbol{R}^1$. A metric space X is *connected* if it is *not* the union of a pair of non-empty open sets which are disjoint. Equivalently, X is connected if and only if the only sets which are both open and closed in X are X itself and the empty set. It can be shown that arc-connectedness implies connectedness. There are examples of connected sets in $\boldsymbol{R}^2$ that are not arc-connected (see Dieudonné). One can show that a subset of the real line is connected if and only if it is an interval. An important property of a continuous function f is that the image of a set S under f, $f(S)$, is connected if S is connected. This result has a very useful implication: Let f be a continuous real-valued function on a connected subset A of $\boldsymbol{R}^m$ such that $f(a_1)<0<f(a_2)$ for some points a_1, $a_2\in A$. Then there exists a solution to the equation $f(x)=0$. To see this, not that $f(A)$ is a connected subset of $\boldsymbol{R}^1$, so $f(A)$ is an interval $[a,b]$. By assumption, $0\in[a,b]$, so there exists $x\in A$ such that $f(x)=0$. A useful corollary of this result is that a pair of real-valued continuous functions g, h on a connected subset A of $\boldsymbol{R}^m$ has a solution to the equation $g(x)=h(x)$ so long as there exists a_1, $a_2\in A$ such that $g(a_1)\geqslant h(a_1)$ and $g(a_2)\leqslant h(a_2)$.

8. Convex sets and cones in Euclidean spaces

Convexity is arguably the most important mathematical property in microeconomics (see Chapters 9–14). Without convexity of preferences, for example, demand and supply functions are not continuous, and so competitive markets generally do not have equilibrium points. The extremely useful results of mathematical programming (see Chapter 2) and duality theory (see Chapter 12) do not hold in the absence of convexity. The economic interpretation of convex production sets is, of course, constant or decreasing returns to scale; the corresponding interpretation of convex indifference curves (or preference sets) in consumer theory is diminishing marginal rates of substitution. Relatively little is known about general economic equilibrium models that allow non-convex production sets (e.g., economics of scale) or non-convex preferences (e.g., the consumer prefers a glass of beer or a glass of champagne alone to any mixture of the two). The only exact results are for economies with an infinite number of participants.

All sets in this section are subsets of $\boldsymbol{R}^m$. A *convex set* $S \subset \boldsymbol{R}^m$ is one which contains a line joining any pair of points belonging to S. More formally, let $x, y \in \boldsymbol{R}^m$. The *line-segment joining* x *and* y, denoted $[x, y]$, is the set of points $z = \alpha x + (1-\alpha)y$ where $0 \leqslant \alpha \leqslant 1$; such a z is called a *convex combination* of x and y. A set S is *convex* if $[x, y] \subset S$ for any pair of points, $x, y \in S$. Any sphere $S_\varepsilon(x)$ is convex. The solution set of any system of linear equations or inequalities is convex.

Some simple properties of convex sets are: (1) $\bar{S}$ is convex if S is convex; (2) S and T convex sets implies $S \cap T$ convex; (3) $S \cup T$ is not necessarily convex even if S and T are convex; (4) $S + T \equiv \{s + t \mid s \in S \text{ and } t \in T\}$ is convex if S and T are convex (thus, the aggregate production possibility set of an economy is convex if each individual firm has a convex production set); (5) let $\lambda \geqslant 0$ be a scalar, then $\lambda S = \{\lambda s \mid s \in S\}$ is convex if S is convex. A very useful topological property of a convex set S is that if S has a non-empty interior then int S is *dense* in $\bar{S}$, i.e., $\bar{S} = \overline{\text{int } S}$. Heuristically, any sphere about $x \in \bar{S}$ has a point in common with int S because of convexity. Thus x is the limit of points belonging to the interior of S.

Another very useful property of convex sets is that any disjoint pair of convex sets can be separated by a hyperplane. This fact is frequently used in economics to obtain a price system that decentralizes a Pareto-efficient allocation of resources, i.e., a price system that leads consumers and producers to choose this allocation. Decentralization is illustrated in Figure 8.1 for a two-good, one-producer, one-consumer economy. Here, Y is the production set, I is the highest indifference curve that shares a point with Y, z is the Pareto-efficient allocation, and H is the separating hyperplane: the set of pairs (x, y) such that $p_x x + p_y y = M$. The sets Y and $P(z)$ are separated by H, where $P(z)$ is the upper preference set bounded by I. When prices are (p_x, p_y) the producer is maximizing profits, and given those profits as income, M, the consumer is maximizing utility subject to his budget constraint.

More formally, a *hyperplane* $H(p, \alpha)$ in $\boldsymbol{R}^m$ is the set of all points $x \in \boldsymbol{R}^m$ satisfying $p \cdot x = \alpha$ for some non-zero vector $p \in \boldsymbol{R}^m$ and some scalar α, i.e., $H(p, \alpha) = \{x \in \boldsymbol{R}^m \mid p \cdot x = \alpha\}$. The vector p is called the *normal* to the hyperplane H (see Figure 8.2).

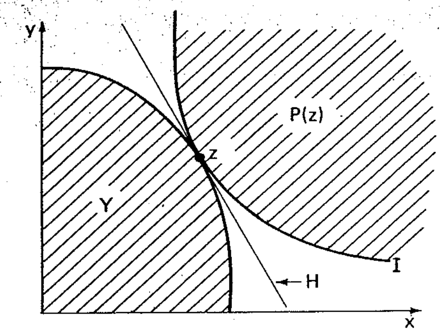

Figure 8.1

A *closed half-space* determined by the hyperplane $H(p, \alpha)$ is either the set of points 'below' or the set of points 'above' H, i.e., either the set $\{x \in \boldsymbol{R}^m \mid p \cdot x \leqslant \alpha\}$ or the set $\{x \in \boldsymbol{R}^m \mid p \cdot x \geqslant \alpha\}$. The set $\{x \in \boldsymbol{R}^m \mid p \cdot x \leqslant \alpha\}$ could be, for example, a consumer's budget set with prices p and income α. An *open half-space* is defined as the interior of a closed half-space. A *bounding hyperplane* $H(p, \alpha)$ to a set S is a hyperplane such that S is completely contained in one of the two closed half-spaces determined by $H(p, \alpha)$; see Figure 8.2. A *supporting hyperplane* $H(p, \alpha)$ for S is a bounding hyperplane that shares a point in common with the boundary of S (more precisely, $\inf\{p \cdot x \mid x \in S\} = \alpha$). A hyperplane $H(p, \alpha)$ *separates* two sets S and T if $p \cdot s \geqslant \alpha$ for all $s \in S$ and $p \cdot t \leqslant \alpha$ for all $t \in T$ (i.e., if S and T are completely contained in opposite closed half-spaces).

We have already given an example of a separating hyperplane above in Figure 8.1. Indeed, that same hyperplane supports the production set Y and $P(z)$, the upper preference set bounded by I, reflecting the fact that the producer is maximizing profits and the consumer is minimizing the cost of obtaining the utility level corresponding to I (and hence, in this case, maximizing utility subject to a budget constraint).

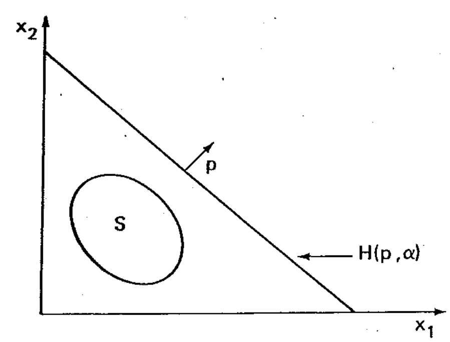

Figure 8.2

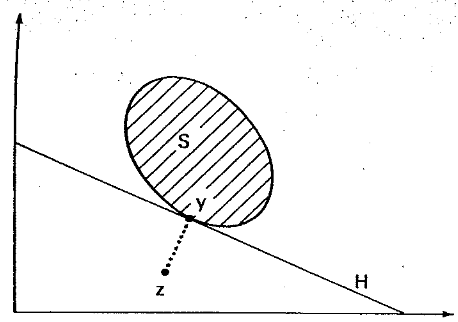

Figure 8.3

Lemma

If S is convex and if $z \notin \bar{S}$, then there exists $y \in \bar{S}$ and $p \neq 0$ such that $p \cdot y = \inf\{p \cdot x | x \in S\} > p \cdot z$ (i.e., there exists a supporting hyperplane H for S that separates S from the point z). In Figure 8.3, y is the point on the frontier of S that is closest to z.

Sketch of the Proof

Let $L = [z, y]$ denote the line joining z and y. Let $H(p, \alpha)$ be the hyperplane which is perpendicular to L and which passes through y. It is intuitive that H is tangent to S at y (i.e., $p \cdot s \geqslant p \cdot y$ for all $s \in S$ in some neighborhood of y). We claim that $p \cdot s \geqslant p \cdot y$ for all $s \in S$, not just in the neighborhood of y. Suppose not, i.e., the exists $x^\circ \in S$ such that $p \cdot x^\circ < p \cdot y$. The line segment $[x^\circ, y]$ belongs to S by convexity and $p \cdot x < p \cdot y$ for all $x \in [x^\circ, y]$ except when $x = y$ (since $p \cdot x = p \cdot (\theta x^\circ + (1-\theta)y) < \theta p \cdot y + (1-\theta) p \cdot y$). But this contradicts the tangency of H to S, since x can be chosen to be in an arbitrarily small neighborhood of y. ■

Supporting Hyperplane Theorem

If y is on the boundary of a convex set S, then there exists a supporting hyperplane for S that passes through y.

Minkowski Separating Hyperplane Theorem

If S and T are convex sets with disjoint interiors, then there exists a hyperplane that separates S and T.

Proof

This follows from the preceding Lemma. For simplicity, take the case where $S \cap T = \phi$. Let $S - T = \{s - t | s \in S, t \in T\}$. Then $0 \notin S - T$. If zero is on the boundary of $S - T$ then there exists a supporting hyperplane $H(p, \alpha)$ for $S - T$

that passes through zero. If not, $0 \notin \overline{S-T}$, so zero is separated from $S-T$. In either case, $0=p\cdot 0 \leqslant p\cdot(s-t)$ for $s\in S$ and $t\in T$, so that $ps \leqslant pt$, as was asserted. ■

An immediate corollary of the Minkowski Theorem is that if Z is convex and disjoint from the non-negative orthant, then there is a separating hyperplane $H(p,\alpha)$ for Z and the non-negative orthant with $p>0$ (i.e., $p\neq 0$ and $p_i \geqslant 0$ for all $i=1,\ldots,m$) such that $p\cdot z \leqslant 0$ for all $z\in Z$. This result can be used to establish the existence of a "decentralizing" price system for Pareto efficient allocations in a multigood, multiperson economic system. See Figure 8.1 and the discussion following it for the basic idea.

A *cone* C in R^m is a set such that if $x\in C$ then $\lambda x\in C$ for any $\lambda \geqslant 0$. A *convex cone* is a cone that is also a convex set. It is easy to see that a convex cone C contains the sum of any two vectors in C and that $0\in C$. An *extreme point* of a convex set S is any point $z\in S$ such that z cannot be expressed as a convex combination of points in S that are distinct from z. The vertices of a triangle in $\boldsymbol{R}^2$ are extreme points, for example. A convex cone is *pointed* if zero is an extreme point of C. A convex production set that contains 0 and satisfies constant returns and irreversibility is a pointed convex cone. It turns out that the sum of convex cones is a convex cone. The union of cones is also a cone (but the union of convex cones is not generally a convex set). The *dual* (or polar cone) of a convex cone C is the set $C^*=\{p\in R^m \mid p\cdot x\leqslant 0 \text{ for all } x\in C\}$. The dual is a convex cone.

Duality Theorem

If C is a closed convex cone, then the dual of the dual of C is C itself; i.e., $(C^*)^*=C$.

The *convex hull* of a set S, denoted con S, is the set of all convex combinations of points in S, i.e., $\operatorname{con} S=\{x \mid x=\sum_{i=1}^n \alpha_i s_i$ for some numbers α_i, $0\leqslant \alpha_i \leqslant 1$, $\sum_{i=1}^n \alpha_i=1$, and some set of points $s_i\in S\}$. Thus, the convex hull of a convex set S is S itself. The convex hull of any set S is a convex set. The convex hull of a pair of points $\{x,y\}$ is the line segment $[x,y]$.

Caratheodory's Theorem

Any point in con S where S is any subset of $\boldsymbol{R}^m$, is the convex combination of at most $m+1$ points of S.

A useful property of convex hulls is that $\operatorname{con}(\sum_{i=1}^n S_i)=\sum_{i=1}^n \operatorname{con}(S_i)$. Although the convex hull of a closed set is not necessarily closed, the convex hull of a compact set is itself a compact set.

Krein–Milman Theorem

Any compact convex set is equal to the convex hull of its extreme points.

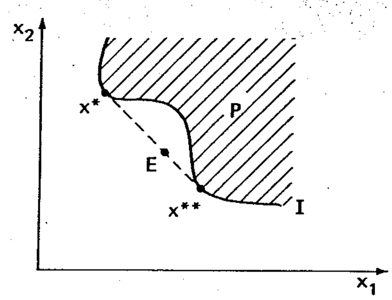

Figure 8.4

Although not much is known about exact solutions of descriptive general equilibrium models when convexity is absent, some important approximation results are known. The *Shapley–Folkman Theorem* forms the mathematical basis of these results; it discovered as a direct consequence of the posing of the question of non-convexities in economics.

Suppose there are a large number of consumers with identical non-convex preferences in a pure exchange economy. In Figure 8.4, I is a typical indifference curve and P is the corresponding upper preference set. Now, examine the fictitious economy generated by taking as preferences the convex hulls of upper preference sets such as P. Such an economy is convex and so has an equilibrium. Let the point E in the Figure 8.4 be the equilibrium to the representative consumer. Suppose E is, for example, $(302/713)x^* + (411/713)x^{**}$. Then, if there were exactly 713 consumers, one could obtain E on average by placing 302 consumers at x^* and 411 at x^{**}. But this would be a competitive equilibrium in the original economy if prices p^* corresponded to the normal of the line segment $[x^*, x^{**}]$. The argument generalizes when the number of consumers is sufficiently large.

Shapley–Folkman Theorem

Let S_i $(i=1,\dots,n)$ be non-empty subsets of R^m, let $S=\sum_{i=1}^n S_i$, and let x be an element of the set $\operatorname{con} S$. Then there exist elements $x_i \in \operatorname{con} S_i$ such that $x=\sum_{i=1}^n x_i$ and $x_i \in S_i$ for all but (at most) m of the sets S_i. Thus, any element in $\operatorname{con} S$ can be expressed as the sum of elements of the S_i, with the exception of at most m elements, no matter how many such S_i there are.

Proof

Because a proof of the Shapley–Folkman Theorem is not readily available, we shall provide one here (due to J. P. Aubin and I. Ekeland). Let $x \in \operatorname{con}(S)$. Caratheodory's theorem guarantees that x can be expressed as the convex combination of a finite number of points, k, of S, $x=\sum_{j=1}^k \alpha_j y_j$ with $y_j \in S$, $\alpha_j > 0$ and $\sum_{j=1}^k \alpha_j = 1$.

Moreover, each y_j may be written as

$$y_j=\sum_{i=1}^{n} y_{ij} \quad \text{with} \quad y_{ij}\in S_i.$$

Let F_i be the set $\{y_{ij}\}_{j=1}^{k}$ consisting of k points. But then $x\in \operatorname{con}(\sum_{i=1}^{n} F_i)$ since $y_j\in \sum_{i=1}^{n} F_i$. As mentioned earlier, the convex hull of a sum of sets is the sum of the convex hulls of those sets, so $\operatorname{con}(\sum F_i)=\sum \operatorname{con} F_i$.

We have replaced each (possibly non-compact) set S_i by a compact (indeed, finite) set F_i. By the Krein–Milman Theorem, any extreme point of $\operatorname{con} F_i$ is an element of F_i (this is not true of non-compact sets!). Consider the set $P=\{(x_i)_{i=1}^{n} \mid x_i\in \operatorname{con} F_i \text{ and } \sum_{i=1}^{n} x_i=x\}$. This compact convex subset of $\boldsymbol{R}^{mn}$ is non-empty since $x\in \sum \operatorname{con} F_i$. Let $(\bar{x}_i)_{i=1}^{n}$ be an extreme point of P. Again, by the Krein–Milman Theorem, $(\bar{x}_i)_{i=1}^{n}$ is an element of P, so $x=\sum_{i=1}^{n} \bar{x}_i$ with $\bar{x}_i\in \operatorname{con} F_i$. We now claim that all but m (at most) of the $\bar{x}_i$ are extreme points of $\operatorname{con} F_i$. Since extreme points of the convex hull of the F_i are members of F_i, we will have proved the theorem.

Suppose not, i.e., there exist $m+1$ of the $\bar{x}_i$ which are not extreme points of $\operatorname{con} F_i$. Suppose these points are $\bar{x}_1,\ldots,\bar{x}_{m+1}$ after relabeling. For each non-extreme $\bar{x}_i$, there exists a non-zero vector $z_i\in \boldsymbol{R}^m$ and a number $\varepsilon_i>0$ such that

(1) for all t, $|t|<\varepsilon_i$, we have $\bar{x}_i+tz_i\in \operatorname{con} F_i$.

Define $\varepsilon=\min\{\varepsilon_i \mid 1\leqslant i\leqslant m+1\}$. We have $m+1$ vectors z_i in m-dimensional space. Thus, they are linearly dependent, i.e., there exist numbers β_i ($i=1,\ldots, m+1$) not all zero, such that $\sum_{i=1}^{m+1}\beta_i z_i=0$. We may assume $|\beta_i|\leqslant 1$.

Next, construct two points of $\boldsymbol{R}^{mn}$, $(x_i')_{i=1}^{n}$ and $(x_i'')_{i=1}^{n}$, as follows:

$$\begin{aligned} x_i'&=\bar{x}_i+\varepsilon\beta_i z_i && \text{for} \quad 1\leqslant i\leqslant m+1,\\ x_i''&=\bar{x}_i-\varepsilon\beta_i z_i && \text{for} \quad 1\leqslant i\leqslant m+1,\\ x_i'&=x_i''=x_i && \text{otherwise.} \end{aligned}$$

By (1), x_i' and x_i'' both belong to $\operatorname{con} F_i$. Furthermore, $\sum_{i=1}^{n} x_i'=\sum_{i=1}^{n}\bar{x}_i+\varepsilon\sum_{i=1}^{n}\beta_i z_i=x$ and $\sum_{i=1}^{n} x_i''=\sum_{i=1}^{n}\bar{x}_i-\varepsilon\sum_{i=1}^{n}\beta_i z_i=x$. Thus, $(x_i')_{i=1}^{n}$ and $(x_i'')_{i=1}^{n}$ both belong to P. But $(\bar{x}_i)_{i=1}^{n}=\frac{1}{2}(x_i')_{i=1}^{n}+\frac{1}{2}(x_i'')_{i=1}^{n}$, so $(\bar{x}_i)_{i=1}^{n}$ is not an extreme point of P, a contradiction. ■

We are interested in a corollary to the Shapley–Folkman Theorem which implies, in effect, that the distance between any element of $\operatorname{con} S$ and some element of S is bounded in a fashion that is independent of n, the number of sets in the sum. To this end, define $d(x, Y)$ as the distance between a point x and a set Y, $d(x,Y)=\inf_{y\in Y}\|x-y\|$, where $\|\cdot\|$ is Euclidean distance. Consider the real-valued function $\rho(Y)=\sup_{x\in \text{con } Y} d(x, Y)$. This number is the maximum of the distances between points in the convex hull from the original set, and so the function ρ is a measure of the non-convexity of any set. Also, $\rho(Y)$ is zero if and only if $\bar{Y}$ is convex. [See Heller (1972) for more detail.]

We use this measure to prove the following corollary, which is central to most of the approximate equilibrium results in the literature:

Corollary

Let $S_1,\ldots, S_n$ be subsets of $\boldsymbol{R}^m$ such that $\rho(S_i)\leqslant c$ for all i and some $c>0$. Let x be any element of $\text{con}(\sum_{i=1}^n S_i)$. Then there are elements $a_i\in S_i$, $i=1,\ldots, n$, such that $\|x-\sum_{i=1}^n a_i\|\leqslant mc$.

Proof

As before, define $S=\sum_{i=1}^n S_i$. It is shown first that $\rho(S)\leqslant \sum_{i=1}^n \rho(S_i)$. For $n=1$, this result is trivial. We use induction on n. Let $x\in \text{con } S$ and $S'=\sum_{i=1}^{n-1} S_i$. Define $x'\in \text{con } S'$ and $x_n\in \text{con } S_n$ so that $x=x'+x_n$. Let $b\in S$ and define $b'\in S'$ and $b_n\in S_n$ so that $b=b'+b_n$. Then, $\|x-b\|\leqslant \|x'-b'\|+\|x_n-b_n\|$. Hence,

$$\inf_{b\in S}\|x-b\|\leqslant \|x'-b'\|+\|x_n-b_n\|,$$

$$\inf_{b\in S}\|x-b\|\leqslant \inf_{b'\in S'}\|x'-b'\|+\inf_{b_n\in S_n}\|x_n-b_n\|,$$

$$\begin{aligned}\inf_{b\in S}\|x-b\|&\leqslant \sup_{x'\in \text{con } S'}\ \inf_{b'\in S'}\|x'-b'\|+\sup_{x_n\in \text{con } S_n}\ \inf_{b_n\in S_n}\|x_n-b_n\|\\ &=\rho(S')+\rho(S_n).\end{aligned}$$

Therefore, $\rho(S)\leqslant \rho(S')+\rho(S_n)\leqslant \sum_{i=1}^n \rho(S_i)$, by the induction hypothesis.

Given $x\in \text{con } S$, use the Shapley–Folkman Theorem to obtain a set of n points $\{x_i\}$ and a set of indices I (with at most m elements) such that for $i\in I$, $x_i\in \text{con } S_i$, while for i not in I, $x_i\in S_i$ and $x=\sum_{i=1}^n x_i$.

We know that for each $i\in I$ there is an $a_i\in S_i$ ($i\in I$), such that

$$(2)\quad \|\textstyle\sum_{i\in I}(x_i-a_i)\|\leqslant \rho\left(\sum_{i\in I}S_i\right)\leqslant \sum_{i\in I}\rho(S_i),$$

by the definition of $\rho(\cdot)$, and by the result of the preceding paragraph. For i not in I, define $a_i\equiv x_i$. Therefore, there exist points $a_i\in S_i$ such that

$$\left\|\sum_{i=1}^n x_i-\sum_{i=1}^n a_i\right\|=\left\|\sum_{i\in I} x_i-\sum_{i\in I} a_i\right\|\leqslant mc,$$

because $\rho(S_i)\leqslant c$ and because of inequalities (2), and the fact that I has no more than m elements. ■

The S_i can be taken to be non-convex production sets or preference sets or some combination of them. Thus, in the aggregate, the discrepancy between an allocation in the fictitious economy generated by con S and some allocation in the real economy is bounded in a way that is independent of the number of economic agents. Therefore, the average agent experiences a deviation from intended actions that vanishes in significance as the number of agents goes to infinity.

9. Concave functions, quasi-concave functions, and homothetic functions

A real-valued function on a convex subset S of $\boldsymbol{R}^m$ is *concave* if for any $x^\circ, x^1 \in S$, we have $f(\theta x^\circ + (1-\theta)x^1) \geqslant \theta f(x^\circ) + (1-\theta) f(x^1)$. Geometrically, the graph of the function lies above the line joining any two points on the graph (see Figure 9.1). The concavity of a function is equivalent to the convexity of the set of points lying below the graph of the function (i.e., the set G in Figure 9.1). A function f is *convex* if $(-f)$ is concave.

If a concave function f is differentiable, then one can show that the tangent plane to any point on the graph of f lies above the graph. Formally, if $\partial f(x^\circ)/\partial x$ is the vector of partial derivatives at some arbitrary x°, then $f(x^1) - f(x^\circ) \leqslant (x^1 - x^\circ)\cdot(\partial f(x^\circ)/\partial x)$, for any x^1 in the domain. If f is a function of a real variable, this fact can be seen as follows: suppose for simplicity $x^\circ = 0$ and $f(0) = 0$. Then we must show $f(x^1) \leqslant x^1 \cdot (\partial f(0)/\partial x)$. For this purpose, let $0 < y < x^1$. By concavity,

$$f(y) \geqslant \left[\frac{x^1 - y}{x^1}\right] f(0) + \left[\frac{y}{x^1}\right] f(x^1) = \frac{y}{x^1} f(x^1).$$

Thus $f(y)/y \geqslant f(x^1)/x^1$. Letting $y \to 0$, we find that $\partial f(0)/\partial x \geqslant f(x^1)/x^1$, as was to be shown.

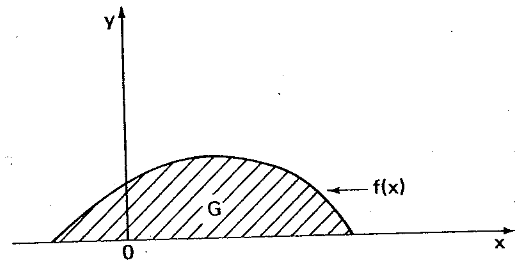

Figure 9.1

This preceding fact shows that the first-order conditions for a maximum, viz, $\partial f(x^\circ)/\partial x=0$, are sufficient for a global maximum at x°, in the event that f is concave. For, then $f(x^1)-f(x^\circ)\leqslant 0\cdot(x^1-x^\circ)=0$. [See Chapter 2 for more detail.]

A function f is *quasi-concave* if for all $\alpha\in R$, the set $\{x\,|\,f(x)\geqslant\alpha\}$ is convex. For example, if f is a utility function that has convex upper preference sets, then f is quasi-concave. A concave function is clearly quasi-concave. A point that satisfies the first-order Lagrangian conditions (see Chapter 2) is a global constrained maximum if the objective function f is quasi-concave and the constraint set is described by concave functions $g_i(x)\leqslant 0$ (so long as a weak technical condition is also satisfied).

A function f is *quasi-convex* if $(-f)$ is quasi-concave. The above results for quasi-concave functions can be recast, of course, for quasi-convex functions with the appropriate changes of inequalities and with the word "minimum" replacing "maximum".

We say that a function f: $X\to\boldsymbol{R}$, where $X\subset\boldsymbol{R}^m$ is *homogeneous of degree k* if, for any $x\in X$ and $\lambda>0$, $f(\lambda x)=\lambda^k f(x)$. A production function that is homogeneous of degree 1 displays constant returns to scale. One can show that the expansion path for a homogeneous function is a ray through the origin: formally, that $\partial f(\lambda x)/\partial(\lambda x_i)=\lambda^{k-1}(\partial f(x)/\partial x_i)$. The expansion path in production theory is the set of input combinations that minimize cost (subject to a given output level) as the level of output expands (cf. Figure 9.2). Thus, the marginal rate of substitution $\mathrm{d}x_2/\mathrm{d}x_1|_{f=\text{const}}$ is unchanging along any ray from the origin. A *homothetic function h* is a monotone increasing transform of a homogeneous (of positive degree) function: $h(x)=g(f(x))$ where g is a monotone increasing function of a real variable and f is homogeneous of some degree $k>0$. A homothetic function also has the property that the marginal rate of substitution is constant along any ray from the origin. In fact, this latter property is equivalent to homotheticity.

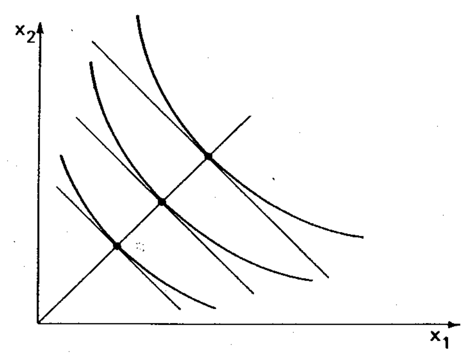

Figure 9.2

10. Hemi-continuity of correspondences and the maximum theorem

A correspondence is a point-to-set function. Formally speaking, let X and Y be any sets. A *correspondence* γ from X into Y is a relation which associates to every element $x \in X$ a unique non-empty subset $\gamma(x)$ of Y. Let 2^Y denote the set of all subsets of Y. Then, in functional notation, we can express the correspondence γ between X and Y as γ: $X \to 2^Y$ with $\gamma(x) \neq \phi$.

Correspondences are important in economics because, for example, there is frequently more than one solution point to a constrained maximum problem. Thus, in the absence of strict convexity of preferences, economists must deal with demand correspondences rather than demand functions.

The *graph* of the correspondence γ: $X \to 2^Y$ is the set $G(\gamma) = \{(x, y) | y \in \gamma(x)\}$. On the other hand, every set S in $X \times Y$ defines a relation ψ as follows: $\psi(x) = \{y | (x, y) \in S\}$; ψ will be a correspondence if $\psi(x) \neq \phi$ for all $x \in X$. The *sum of correspondences* γ_i: $X \to 2^Y$ $(i = 1, \ldots, n)$, where Y is a Euclidean space, is defined as $\sum_{i=1}^n \gamma_i(x) \equiv \{y \in Y |$ there exists $y_i \in \gamma_i(x)$, for all i, such that $y = \sum_{i=1}^n y_i\}$. The *cross-product of correspondences* γ_i: $X \to 2^Y$ $(i = 1, \ldots, n)$ is defined as $\gamma_1(x) \times \gamma_2(x) \times \cdots \times \gamma_n(x) \equiv \{(y_1, y_2, \ldots, y_n) | y_i \in \gamma_i(x)$ for each $i\}$. The *composition of two correspondences* γ: $Y \to 2^Z$ and ψ: $X \to 2^Y$ is defined as $\gamma \circ \psi(x) \equiv \{z |$ there exists $y \in \psi(x)$ such that $z \in \gamma(y)\}$.

For the rest of this section, suppose that X and Y are topological spaces. Hemi-continuity of correspondences is an essential property for obtaining the existence of fixed points. A correspondence γ is *upper hemi-continuous* (abbreviated *u.h.c.*) *at* $x° \in X$ if for any open set V which contains all of $\gamma(x°)$, there exists a neighborhood $U(x°)$ such that $\gamma(x) \subset V$ for all $x \in U(x°)$. The correspondence γ is *upper hemi-continuous* if it is u.h.c. at every $x \in X$. A correspondence which is *not* u.h.c. at $x°$ "blows up" in any neighborhood of $x°$ in the sense that part of $\gamma(x)$ lies outside some small open set containing $\gamma(x°)$, as occurs in Figure 10.1.

It is easy to show that if γ is point valued (i.e., γ is a function), then γ is u.h.c. at $x°$ if and only if it is a continuous function at $x°$. The term "upper semi-continuity" for correspondences is frequently used in the literature, rather than upper hemi-continuity. The latter term was recently adopted to avoid conceptual confusion with usage of upper semi-continuity for functions. [A function is upper semi-continuous at $x°$ if for each $\varepsilon > 0$, there exists a neighborhood $N(x°)$ such that $x \in N(x°)$ implies $f(x) < f(x°) + \varepsilon$.] There are numerous examples of functions that are upper semi-continuous but not continuous. We say $\gamma(x)$ is *compact-valued* if $\gamma(x)$ is a compact set for every $x \in X$. A correspondence γ is *closed at* $x°$ if for every sequence $(x_n, y_n) \in G(\gamma)$ such that $(x_n, y_n) \to (x°, y°)$ it is true that $(x°, y°) \in G(\gamma)$, so that if $y^n \in \gamma(x^n)$ and $x^n \to x°$ and $y^n \to y°$, then $y° \in \gamma(x°)$. A correspondence is *closed* if it is closed at each $x°$. It is immediate that γ is closed if and only if $G(\gamma)$ is a closed set. Let the *image of a*

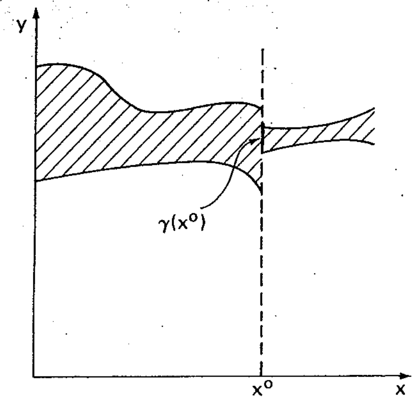

Figure 10.1

t $K\subset X$ *under* γ be defined as $\gamma(K)=\{y\,|\,y\in\gamma(x)$ for some $x\in K\}$. The *range* f γ is $\gamma(X)$. Suppose that the range of γ is compact and $\gamma(x)$ is a closed set for ach x. It then turns out that γ is closed if and only if it is u.h.c. The latter result frequently useful in establishing u.h.c. However, not all closed correspon-ences are u.h.c., even when compact-valued. Let γ: $\boldsymbol{R}\to 2^{\boldsymbol{R}}$ be such that

$$\begin{aligned}\gamma(x)&=\{0\}, \qquad x=0,\\ &=\{1/x\}, \quad x>0.\end{aligned}$$

he following condition is sufficient for u.h.c. at x° when γ is compact-valued nd X and Y are metric spaces: For every pair of sequences (x^n), (y^n) such that $^n\to x^\circ$ and $y^n\in\gamma(x^n)$, there is a convergent subsequence of (y^n) whose limit elongs to $\gamma(x^\circ)$.

Let γ: $X\to 2^Y$, where X and Y are metric spaces; the following properties of .h.c. correspondences are useful:

1) $\overline{\gamma(x)}$ is u.h.c. if γ is u.h.c.
2) $\cup_{i=1}^n\gamma_i$ is u.h.c. if the γ_i are u.h.c.
3) If $(\gamma_i)_{i=1}^n$ are all u.h.c. and compact-valued, then $\sum_{i=1}^n\gamma_i$ is u.h.c. and compact-valued, where Y is a Euclidean space.
4) The cross-product of u.h.c. and compact-valued correspondences is u.h.c. and compact-valued.
5) If γ is u.h.c. and compact-valued, then the convex hull correspondence $\operatorname{con}\gamma$ (defined by $\operatorname{con}\gamma(x)\equiv\operatorname{con}[\gamma(x)]$) is also u.h.c. and compact-valued, when Y is a subset of Euclidean space.
6) If K is a compact set and if γ is u.h.c. and compact-valued, the image of K under γ, $\gamma(K)$, is a compact set.

(7) The composition $\gamma \circ \psi$ of two u.h.c. correspondences is u.h.c.
(8) If γ_1 and γ_2 are u.h.c. and closed-valued, then $\gamma_1 \cap \gamma_2$ is u.h.c. if $\gamma_1 \cap \gamma_2(x) \neq \phi$ for all $x \in X$.

A correspondence is *lower hemi-continuous* (or *l.h.c.*) *at* x°, if for every open set V that meets $\gamma(x^\circ)$, i.e., $\gamma(x^\circ) \cap V \neq \phi$, there exists a neighborhood of x°, $U(x^\circ)$, such that $\gamma(x)$ also meets V for every $x \in U(x^\circ)$. We say that γ is *l.h.c.* if it is l.h.c. at every $x \in X$. Geometrically, the idea is that $\gamma(x)$ does not suddenly contract in size if we move slightly away from x°, as occurs in Figure 10.2 (where there is a "spike" at x°). Again, if $\gamma(x)$ is single-valued, then the *correspondence* γ is l.h.c. if and only if the *function* γ is continuous.

When X and Y are metric spaces the following sequence condition is necessary and sufficient for γ to be l.h.c. at x°: for every sequence (x^n) which converges to x° and for every $y^\circ \in \gamma(x^\circ)$, there exists a sequence (y^n) such that $y^n \in \gamma(x^n)$ (for all n) and $y^n \to y^\circ$.

The following is a list of useful properties of l.h.c. correspondences when X and Y are metric spaces:

(1) $\bar{\gamma}$ is l.h.c. if γ is l.h.c.
(2) $\cup_{i=1}^{n} \gamma_i$ is l.h.c. if the γ_i are l.h.c.
(3) The composition $\gamma_1 \circ \gamma_2$ is l.h.c. if each of the γ_i are l.h.c.
(4) The sum of l.h.c. correspondences is an l.h.c. correspondence, if Y is a Euclidean space.
(5) The cross-product of l.h.c. correspondences is l.h.c.
(6) Let Y be a subset of Euclidean space. Then $\operatorname{con} \gamma(x)$ is l.h.c., if γ is l.h.c.
(7) The intersection of two l.h.c. correspondences is *not* in general l.h.c.; however:
(8) Let X and Y be subsets of Euclidean space. If γ_i, $i=1,2$, are two l.h.c. convex-valued correspondences such that $\operatorname{int} \gamma_1(x^\circ) \cap \operatorname{int} \gamma_2(x^\circ) \neq \phi$, then $\gamma_1 \cap \gamma_2$ is l.h.c. at x°.

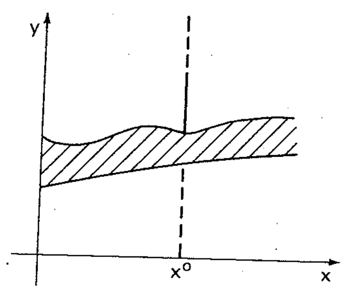

Figure 10.2

correspondence is *continuous at* x° if it is both u.h.c. and l.h.c. at x°. A *inuous correspondence* is continuous at each $x \in X$. The budget correspon-ce $B(p, \omega) = \{x \in C \mid p \cdot (x-\omega) \leqslant 0\}$ is continuous if the endowment vector nt C and C is convex, where C is the set of possible consumption vectors p is the price vector. This follows from property (8) for l.h.c. correspon-ces and from the facts that: (a) C and $\{x \mid p\cdot(x-\omega) \leqslant 0\}$ are convex sets, (b) $\omega - \varepsilon(1, 1, \ldots, 1) \in \operatorname{int} C$ for $\varepsilon > 0$ sufficiently small and $p\cdot(z-\omega) < 0$, since nt $C \cap \{x \mid p\cdot(x-\omega) < 0\}$ in that case, and (c) $\{x \mid p\cdot(x-\omega) \leqslant 0\}$ is continu-s in (p, ω) and $C(p, \omega) \equiv C$ is also continuous in (p, ω).

A *demand correspondence* $x(p, \omega)$ is the set of maximizers of $U(x)$ subject to $B(p, \omega)$. Of great interest in economics are the continuity properties of $p, \omega)$.

aximum Theorem

et X be a topological space. If F is a continuous, real-valued function of X and is a continuous compact-valued correspondence from Y to subsets of X, then e correspondence γ defined by $\gamma(y) = \{x \in B(y) \mid F(x) \geqslant F(x') \text{ for all } x' \in B(y)\}$ u.h.c. and compact-valued, and the function f defined by $f(y) \equiv F(\gamma(y))$ is a ntinuous function.

Note that if either of the sets X or Y in the statement of the Maximum heorem is a compact set, then γ is a closed correspondence; for if X is mpact, then $\gamma(y) \subset X$ and $\gamma(y)$ compact-valued and u.h.c. implies γ is a closed rrespondence. On the other hand, if Y is compact then $B(Y)$ [i.e., range of (y)] is compact, since the image of a compact set under an u.h.c. correspon-ence is compact, so again $\gamma(y)$ is contained in the same compact set for all $\in Y$.

A sketch of the proof of the first conclusion of the Maximum Theorem may be helpful in grasping its meaning. Suppose, for simplicity, that X and Y are metric spaces and that Y is compact. We shall show that γ is closed, i.e., $y^n \to y^\circ$, $x^n \in \gamma(y^n)$ (for all n) and $x^n \to x^\circ$ imply $x^\circ \in \gamma(y^\circ)$. Since $B(y)$ is u.h.c., $x^\circ \in B(y^\circ)$. Let z° be any element of $B(y^\circ)$. By the l.h.c. of $B(y)$, there exists a sequence $z^n \in B(y^n)$ such that $z^n \to z^\circ$. But $x^n \in \gamma(y^n)$ means that $F(x^n) \geqslant F(z^n)$. The function F is continuous, so $F(x^\circ) \geqslant F(z^\circ)$. But this means precisely that $x^\circ \in \gamma(y^\circ)$, as was to be shown.

11. Fixed point theorems

A *fixed point of a function* f: $X \to Y$ (where $X \cap Y \neq \phi$) is a point $\bar{x} \in X$ such that $\bar{x} = f(\bar{x})$. Fixed point theorems are useful for establishing the existence of solutions to a system of nonlinear equations. In economics, fixed point theorems are most often used to guarantee the existence of equilibrium in a wide variety of models of the economy. For instance, let $f(p)$ be an m-dimensional excess demand function. A vector $\bar{p}$ such that $f(\bar{p})=0$ is a competitive equilibrium, since demand equals supply at $\bar{p}$. A fixed point $\bar{p}$ of the function $f(p)+p$ would therefore be a competitive equilibrium. (Unfortunately, establishing the existence of competitive equilibrium is more complex than the preceding sentence suggests.)

The importance of establishing the existence of solutions to economic models is sometimes underemphasized. It is not uncommon for the researcher to find that an economic model is vacuous in having no solutions. The well-known Cournot duopoly model is a case in point. Except in the unlikely circumstances of identical firms or concave demand functions, there may be no Cournot (pure-strategy) equilibrium [see Roberts and Sonnenschein (1977)].

Brouwer Fixed Point Theorem

If X is a compact convex subset of $\boldsymbol{R}^m$ and $f: X \to X$ is a continuous function, then there is a fixed point.

The Brouwer theorem is relatively easy to establish when f is a continuous function of a real variable. Thus, let $X=[0,1]$. Consider Figure 11.1: If f crosses the 45° line, then f has a fixed point. We might as well assume that $f(0)>0$, since otherwise zero is a fixed point. Suppose f lies above the 45° line everywhere. But this is impossible since $f(1) \leqslant 1$ by the assumption that $f: X \to X$.

A function f: $(S, d) \to (S, d)$ is said to be a *contraction* if there exists $r<1$ such that for all $x, y \in S$, $d(f(x), f(y)) < r\, d(x, y)$. Clearly, any contraction is a continuous function.

Contraction Mapping Theorem

A contraction on a complete metric space has a unique fixed point.

Although no assumptions about convexity or finite dimensionality of S are needed for this theorem, contractions are rather special functions.

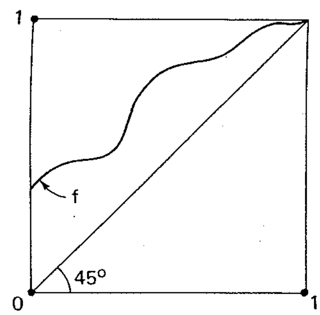

Figure 11.1

A *fixed point of a correspondence* $\gamma(x)$ (where $\gamma: X \to 2^X$) is a point $\bar{x}$ such that $\bar{x} \in \gamma(\bar{x})$.

Kakutani Fixed Point Theorem

Let X be a compact convex subset of R^m, and let γ be a closed correspondence from X into subsets of X. If $\gamma(x)$ is a convex set for every $x \in X$, then there is a fixed point.

An interesting exercise for the reader is to give counter-examples to this result in each circumstance in which any one of the assumptions is not satisfied. For instance, let $\gamma(x) = X \backslash B_\varepsilon(x)$, i.e., $\gamma(x)$ is the complement of an open ball centered on x. Then $\gamma(x)$ is connected, but not convex, γ is closed, and γ has no fixed point.

Reference notes

A good, rigorous survey of some economic applications of the methods discussed here is the textbook by Takayama. His book also includes proofs of many of the less advanced mathematical results. Detailed below are some good expositions of the proofs of the theorems cited in this chapter, as well as their extensions. Many of these sources also have excellent bibliographies (e.g., Arrow–Hahn, Hildenbrand, Klein, Rockafeller, Smart, and Takayama).

Sections 1–6. Good intermediate-level sources are Rudin (1964) and Simmons (1963).

Section 7. A good source is Dieudonné (1960, ch. 3).

Section 8. Arrow–Hahn (1971, app. B), Hildenbrand–Kirman (1976, app. II), Karlin (1959, app. B), Klein (1973, pp. 72–76, 323–341), and Nikaido (1968, pp. 15–44) contain proofs of most of the results of this section. Excellent advanced treatments of convex sets are in Berge (1963, ch. 7 and pp. 158–168) and Rockafeller (1970, pp. 3–22, 43–81, 95–101, 153–212). The first proof of the Shapley–Folkman Theorem appeared in Starr (1969). Artstein (1976) has an interesting general result that implies both the Carathéodory and the Shapley–Folkman Theorems, among others.

Section 9. Nikaido (1968, pp. 44–53) is a good source for proofs of most of the results in this section. Rockafeller is the classic treatise on convex functions.

Section 10. Hildenbrand–Kirman (1976, app. III) and Klein (1973, ch. 6) are good elementary sources. Berge (1963, ch. 6) and Nikaido (1968, pp. 70–73) are good intermediate-level sources. Hildenbrand (1974, pp. 21–35) and Heller (1978) contain useful results for economics about the continuity properties of correspondences.

Section 11. Good elementary proofs of Brouwer's Theorem are in Burger (1963, app.), Klein (1973, ch. 7) and Tompkins (1964). Kakutani's Theorem is also proved by Burger and by Klein. More advanced treatments of the two theorems are given in Berge (1963, pp. 168–176) and Nikaido (1968, pp. 53–70). Arrow–Hahn (1971, app. C) contains proofs of the Brouwer and Kakutani Theorems based on Scarf's algorithm. A good advanced reference for a wide variety of fixed point theorems is Smart.