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A Sampling Approach to the Free Rider Problem

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6.1. Introduction

The problem of ascertaining the preferences of a group of individuals for a public good, or for governmental action in general, can be thought of as a special case of decision theory under imperfect information. The tastes of each individual are privately held pieces of data. In order to insure that an optimal action will be taken, the government must be able to elicit them without cost and without error. One cannot assume that individuals will correctly report their tastes if asked. Methods must be devised under which the correct elicitation of preferences is to the advantage of individuals in the context of the decision making mechanism.

In Green and Laffont [1977] we described the methods by which the true preferences would be willingly revealed by each individual independently of his beliefs concerning the true or professed preferences of others. The basic idea, originally due to Vickrey (1961), Clarke (1971) and Groves (1973) can be stated as follows:

Individuals are assumed to have preferences described by the following utility functions:

For each $i = 1, \dots, n$:

$$\begin{aligned} u_i &= x_i + v_i && \text{if the public project is accepted} \\ &= x_i && \text{if the public project is rejected} \end{aligned}$$

where x_i is the amount of a transferable private good consumed by the individual i and v_i his true willingness to pay.

Let w_i be the agent i 's stated willingness to pay, $i = 1, \dots, n$:

A mechanism is a mapping, f ,

$$f: R^n \rightarrow \{0, 1\} \times R^n$$

written as $f(w_1, \dots, w_n) = (d, t_1, \dots, t_n)$ where $d \in \{0, 1\}$ is the public decision taken (1 = accept project; 0 = reject project) and t_i is the transfer of the private good received by agent i .

The mechanisms defined by

$$d = 1 \quad \text{iff} \quad \sum_i w_i \geq 0$$

$$\begin{aligned} t_i &= \sum_{j \neq i} w_j + h_i(w_{-i}) & \text{if } d = 1 \\ &= h_i(w_{-i}) & \text{if } d = 0 \end{aligned}$$

where $w_{-i} = (w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_n)$ and $h_i(\cdot)$ are arbitrary real-valued functions, can be shown to have the property that setting $w_i = v_i$ is a dominant strategy for each agent, and therefore that a Pareto optimal outcome will always be found. Moreover, as we showed (Green and Laffont, 1977), this class of mechanisms exhausts all those with this attractive property.

If this mechanism could be costlessly applied, the "free rider" problem could be solved in this way—at least for environments with separable utility functions.^a

However, as we will argue below, the use of such mechanisms entails auxiliary costs. It may therefore not be optimal to use these mechanisms, or to use them for eliciting the tastes of the entire population.

Ideally, the preferences of each individual should be accurately elicited before a public decision is taken. This is generally impossible, or undesirable, for two reasons. There may be costs of inquiring itself, and there may be problems in ascertaining individuals' true preferences once they have been solicited. Realistic decision making procedures must try to avoid these obstacles as much as possible, while taking the costs of doing so into account.

Typically, the number of individuals potentially affected by the public project is large—and this fact creates both problems and opportunities in the design of preference elicitation and decision making mechanisms.

Because individuals are numerous, it is natural to attempt to economize on information costs through random sampling. The sampled value, to be an accurate proxy for the tastes of the population, must be based on honest responses by individuals and must include a representative, unbiased selection of them. In a noncoercive society, one cannot, presumably, force some individuals to participate in the sampling process while others are left out. Thus, in addition to arranging the mechanisms for accurate elicitation of preferences and Pareto optimal decision making, the condition of universal willingness to participate must be regarded as an additional desideratum. In this way a purely random sample will be representative of the population.

The remainder of this paper is therefore organized as follows. In section 6.2

^aIn Green and Laffont (1977) we show that there are no mechanisms having this dominant strategy property when the class of allowable preference relations is extended beyond this. Problems with consumption sets for the private good that are bounded below can be easily handled however, see Green and Laffont (1976c), but some modification of this characterization theorem must be made.

we introduce the various reasons why drawing a random sample from the population might be desirable. This can be broken down into two major subheadings: First, (section 6.2.1) we consider random sampling procedures in which one of the mechanisms mentioned above is used to elicit the preferences of the sample precisely. Here the goal of sampling is to balance the associated costs, which are often increasing in the sample size with the risk that sampling variance will cause a nonoptimal decision to be taken. Second (section 6.2.2), we consider the cases in which a random sample from the population is taken in order to improve the accuracy of the responses given. In these cases we will ignore the associated costs mentioned above. Sampling will be designed to balance the accuracy of answers obtained against the sampling variance. Then, we see in which sense sampling can prevent coalitions from distorting their responses. Finally, in section 6.3 we take up the issue of participation and obtain the mechanisms which induce universal participation. The requirement of universal participation is then shown to be incompatible with even weak individual rationality.

6.2 The Use of Sampling to Improve Preference Revelation Mechanisms

6.2.1 Sampling to Reduce Direct Costs of Mechanisms

In this section we will consider the potential value of taking a random sample from the population, eliciting their tastes via a dominant-strategy mechanism as above, and making the decision for the entire group on the assumption that the sample's preferences are representative. We will be concerned with two questions in this regard: First we investigate properties of the optimal sample size. Second, we study whether and to what extent the resulting mechanism, whose outcomes and costs may be regarded as random functions of the true preferences, can be regarded as a solution to the free rider problem. More specifically, we compare the expected value of using this procedure, with the optimal sample size, to having perfect information or to being constrained to make the decision on the basis of the *a priori* beliefs of the planner. To the extent the imperfectness of information concerning tastes can be overcome by these methods, we can say that progress towards the solution of the free-rider problem has been made.

In this subsection we consider two obvious types of costs associated with the use of the dominant-strategy mechanisms. First, the process of elicitation of responses and their compilation might involve the expenditure of real resources. Letting N be the population size and n be the sample size, we assume that these costs are proportional to the latter, that is:

$$C = cn \quad (6.1)$$

where C is the direct cost of the mechanism.

A second type of cost is that which is due to the income transfers induced by the procedure. If we assume that the procedure is used on a population for which the existing income distribution is optimal, then a bound on the expected costs of the income transfers induced will be

$$C = \sum_{i=1}^n |t_i(w)| \quad (6.2)$$

These costs depend on the sample size both because the transfers are only distributed (collected) within the sample group and because the transfers received by each individual are dependent on the statements made by all others, and will vary according to the number of other agents in the sample.

The benefit of sampling is that we can reduce the risk of accepting or rejecting the project when the opposite action would have been superior. To quantify this value, it is necessary to suppose the planner has prior beliefs concerning the preferences of the population which will be modified in light of the sample. For mathematical simplicity we will suppose that he believes that the N individuals' willingness to pay, v_i , are independently and identically normally distributed

$$v_i \sim N(m, s^2)$$

Moreover he is uncertain about the mean, m , but (for simplicity) knows the variance. His prior on this parameter is described by

$$m \sim N(\mu, \sigma^2)$$

When a sample of size n is taken, the planner revises his beliefs about m according to Bayes theorem.

Letting $w = \sum w_i/n$ be the mean of the sample's responses, we have that the posterior distribution is described by

$$m \sim N\left(\frac{s^2\mu + n\sigma^2w}{s^2 + n\sigma^2}, \frac{\sigma^2s^2}{s^2 + n\sigma^2}\right)$$

We will assume throughout that the planner is risk neutral. Thus the expected value of constructing the project is $(\mu s^2 + n\sigma^2w)/(s^2 + n\sigma^2)$.

Recall that the rules under which the mechanisms operate are that the project is constructed only if the stated responses indicate that it is valuable. That is, in the process of eliciting the true preferences, the planner announces the mechanism to be used—including the rule for project acceptance,

$$\sum_{i=1}^n w_i \geq 0$$

It is the spirit of this analysis that the mechanism is then followed, even though this might not be optimal ex post, as would be the case if μ and w were of opposite signs but

$$|\mu| > \frac{n\sigma^2}{s^2} |w|$$

Thus, the expected value of adopting the project for each of the $(N - n)$ individuals who were not sampled is given by:

$$\int_0^\infty \left(\frac{s^2\mu + n\sigma^2 w}{s^2 + n\sigma^2} \right) f(w) dw$$

where $f(w)$ is the ex ante probability law of w :

$$w \sim N\left(\mu, \frac{s^2}{n} + \sigma^2\right)$$

For the sampled group, however, there is no residual uncertainty about their evaluations and, per capita, their expected willingness to pay for the decision that their responses induce is:

$$\int_0^\infty w f(w) dw$$

Thus the ex ante expected evaluation of the decision, when a sample of size n is drawn from a population of size N is

$$\int_0^\infty \left[(N - n) \frac{s^2\mu + n\sigma^2 w}{s^2 + n\sigma^2} + nw \right] f(w) dw$$

This can be evaluated (see Green and Laffont (1974, appendix)) as:

$$\frac{N\sigma^2 + s^2}{\sqrt{2\pi}(s^2/n + \sigma^2)^{1/2}} \cdot e^{-\frac{\mu^2}{2(s^2/n + \sigma^2)}} + \mu N \rho\left(\frac{\mu}{(s^2/n + \sigma^2)^{1/2}}\right)$$

where

$$\rho(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-z^2/2} dz$$

We consider here only the case in which $\mu = 0$ (evaluation of the objective function when $\mu \neq 0$ is carried out in Green and Laffont (1974), by analogous methods). Thus the expected value of the decision becomes

$$\frac{N\sigma^2 + s^2}{\sqrt{2\pi}(s^2/n + \sigma^2)^{1/2}} \quad (6.3)$$

6.2.1.a. Sampling Cost Constant per Capita. We now consider the case of constant per capita sampling cost (6.1) and derive the optimal sample size. Let $V^I(n, N)$ be the ex ante expected evaluation for this problem, which is (6.3) minus (6.1).

$$V^I(N, n) = \frac{(N\sigma^2 + s^2) - cn\sqrt{2\pi}\left(\frac{s^2}{n} + \sigma^2\right)^{1/2}}{\sqrt{2\pi}\left(\frac{s^2}{n} + \sigma^2\right)^{1/2}} \quad (6.4)$$

Taking the derivative of this with respect to n :

$$\frac{dV^I(n, N)}{dn} = \left(\frac{N\sigma^2 + s^2}{2}\right)\frac{s^2}{n^2} - c\sqrt{2\pi}\left(\frac{s^2}{n} + 2\right)^{3/2} \quad (6.5)$$

We are interested in computing the rate of growth of the optimal sample size. Therefore we set $n = N^\delta$ and look for the value of δ such that (6.5) is zero for large N . It is easy to show that the positive terms in (6.5) are increasing at the rate of $N^{1-2\delta}$, whereas all of the negative terms are nonincreasing as functions of N . Therefore, $\delta = 1/2$ is required for the first order condition. The sample size at the optimum will be proportional to the square root of the population size for large economies.

At this point we should pause to consider this somewhat puzzling result. Sampling theory teaches us that the accuracy of a sample is independent of the population size. One might therefore expect a bounded sample size to be optimal, rather than one growing like the square root of the total population. However, as the population size increases, a given degree of precision is no longer optimal. Any error made now affects a larger group of unsampled agents. Therefore a higher degree of precision becomes necessary.

6.2.1.b. Sampling Costs Proportional to Total Transfer Payments. We now neglect the cost of resources used in the sampling process and introduce the cost of distortions induced through the transfers mandated by the preference revelation mechanism. These transfers will, of course, depend upon the particular member of the class of preference revelation mechanisms being employed. In

this section we consider two specific mechanisms; our choice is motivated by the results concerning the inducement of participation in the process presented in section 6.3. These mechanisms are defined by

$$h_i(w_{-i}) \equiv 0 \quad \text{for all } i$$

and

$$h_i(w_{-i}) = \min(-\sum_{j \neq i} w_j, 0) \quad \text{for all } i$$

(This second mechanism is referred to as the pivotal mechanism (see Green and Laffont, 1976).)

For the mechanism $h \equiv 0$, the transfer payments are $n(n-1)w$ if $w \geq 0$ and zero otherwise.

As a limiting hypothesis we regard the entire volume of these payments as a dead-weight loss. This is a very pessimistic viewpoint since we will be neglecting their beneficial effects on the welfare of the sample while fully counting the cost that their collection will impose on the unsampled group.

Thus the ex ante value of the procedure, defined as $V^{\text{II}}(n, N)$ is

$$V^{\text{II}}(n, N) = \frac{(N + n - n^2) \sigma^2 + (2 - n)s^2}{\sqrt{2\pi}(\frac{s^2}{n} + \sigma^2)^{1/2}}$$

Differentiating with respect to n and setting $n = N^\delta$ we can show that the derivative will be asymptotically negative whenever $\delta > 1/3$. Thus the largest local maximum of the objective function occurs when the sample size grows like the cube root of the population size (see Green and Laffont, 1974, for further details). This problem is not a concave one, however, and we cannot rule out zeros at smaller values of δ on purely theoretical grounds. Our computer simulations have shown, however, that over a wide range of the parameter values the zero of the order of the cube root of N is the global maximum.

Further analysis of this problem is necessary in order to show that the procedure is of positive information value. By this we mean that its optimal value exceeds what could be attained through the naive strategy that would accept or reject the project according to the sign of μ . This has an ex ante value of

$$N \max(\mu, 0)$$

since no transfers are required. The information value of the procedure depends on the parameters of the problem σ^2 , s^2 and μ , as well as N and n . Let G be the informational value of the mechanism for a sample of size n .

$$G(n, N; \sigma^2, s^2, \mu) = V^{\text{II}}(n, N) - N \max(\mu, 0)$$

The following results can be demonstrated by using the approximation $n^*(N) = N^{1/3}$ which therefore provides a lower bound on G :

- (1) For $\mu = 0$, G is always nonnegative.
- (2) For $\mu \neq 0$, G may be negative for some N .
- (3) However, $\lim_{N \rightarrow \infty} G(n^*(N), N; \sigma^2, s^2, \mu)$ is always nonnegative.

Moreover if $\sigma^2 \neq 0$ so that the decision problem is nontrivial, the above nonnegativity results can be converted into strict positivity.

Thus, in summary, the costs of this procedure, which grow like the square of the sample size, but only accrue when the project is accepted, account for the lower rate of growth of the optimal sample size than was derived in the constant per capita cost case. However, these costs are not so high as to cause us to revert to the naïve strategy. Nevertheless it would be incorrect to say that a complete solution to the free rider problem has been found—for it is easy to see that the value of the procedure does not approach that of perfect information, which is

$$\max\left(\sum_{i=1}^N v_i, 0\right)$$

In order to improve upon the solution given above we investigate the mechanism defined by

$$h = \min\left(-\sum_{j \neq i} w_j, 0\right)$$

This mechanism has the property that an individual is taxed whenever the social decision is altered on account of his statement. In this case he pays $|\sum_{j \neq i} w_j|$ since this is the cost that his participation and revealed preferences has imposed on the other members of the group. Note however that in those cases in which the project is overwhelmingly carried $-\sum_i w_i$ is larger than any of individual w_i —no taxes will be paid by anyone. This is to be contrasted with the situation for the $h = 0$ mechanism which would produce a total transfer of $(n-1) \sum_{i=1}^N w_i$.

Although it is not true that the

$$h = \min\left(-\sum_{j \neq i} w_j, 0\right)$$

mechanism involves less transfer payments in every case than $h = 0$ would,^b it is

^bConsider the three-person situation: $v_1 = +9$; $v_2 = -5$; $v_3 = -5$. A straightforward calculation reveals that under $h = \min\left(-\sum_{j \neq i} w_j, 0\right)$ the transfers received are: $t_1 = 0$, $t_2 = -4$, $t_3 = -4$, so that $\sum |t_i| = 8$; while under $h = 0$ they are $t_1 = 0$, $t_2 = 0$, $t_3 = 0$, so that $\sum |t_i| = 0$.

clear in an intuitive sense that the average payment would be lower in some statistical sense. This is made precise by the following theorem:

Theorem 6.2.1 (Green and Laffont, 1976b)

Let v_i be independently identically distributed according to $f(\cdot)$, a distribution with finite variance. Then the mechanism defined by

$$h_i(w_{-i}) = \min \left(- \sum_{j \neq i} w_j, 0 \right)$$

produces transfers t_i , $i = 1, \dots, n$ always nonpositive such that $E \sum_{i=1}^N t_i / \sqrt{N}$ approaches a finite limit as $N \rightarrow \infty$.

Following our earlier procedure, we can define the ex ante expected value of using this mechanism on a sample of size n . In the normal case one can calculate

$$E_n = E \sum_{i=1}^n t_i(w_{-i})$$

exactly, not just its limit. This is (see Green and Laffont, 1976b):

$$\begin{aligned} E_n &= \frac{s^2}{\sqrt{2\pi}} [n\sqrt{n-1} - (n-1)\sqrt{n}] \\ &\approx \frac{s^2}{\sqrt{2\pi}} \left[\frac{\sqrt{n}}{2} - \frac{1}{8\sqrt{n}} \right] \dots \end{aligned}$$

The ex ante value of the mechanism is therefore

$$V^{\text{III}}(n, N) = \frac{(N-n)\sigma^2}{\sqrt{2\pi} \left(\frac{s^2}{n} + \sigma^2 \right)^{1/2}} - E_n$$

Calculations analogous to the ones described above give the order of the optimal sample size, namely

$$\frac{n^*(N)}{N^{2/3}} \rightarrow \text{constant}$$

The expected transfer costs are much smaller with the pivotal mechanism than with the mechanism associated to $h \equiv 0$. Consequently the optimal sample size is much higher.

Indeed, if we introduced simultaneously per capita sample costs and transfer costs with the pivotal mechanism, the per capita sample costs would dominate in the determination of the optimal sample size.

6.2.2. Sampling to Improve the Accuracy of Responses

When any of the dominant strategy mechanisms are used, the individuals sampled will respond with their true willingness to pay provided they act rationally and as individuals. However, we may expect to have difficulties with eliciting accurate responses if the individuals do not know their own willingness to pay precisely, or if they perceive that they can collude with others, jointly distorting their preferences.

6.2.2.a. The Strength of Incentives. Since the mechanisms may be used to decide on undertaking new public projects whose exact characteristics are unobservable *ex ante*, individuals may justifiably be uncertain about what their own *ex post* willingness to pay for the project are. The mechanism will thus bear a double burden. It should encourage individuals to ascertain their own true preferences and also to reveal them for use in the decision making process. What is the role of sampling in furthering these ends? It is a universal problem for any decision process concerning large groups that the strength of the individual's incentive is low. In majority rule the prospect that one's vote will be decisive is very minor.

In the process above, similarly, the utility of an individual is determined by the decision and by everyone else's stated willingness to pay. Thus since the probability that one's statement will change the sign of the aggregate is slight, there is little incentive to refine personal information about the true willingness to pay. In the language of statistical decision theory the regret function has its minimum at the truth, but it is very flat. If acquiring this exact information is costly, an individual may choose to respond to the mechanism on the basis of his prior beliefs alone. The decision is more likely to be nonoptimal if a larger proportion of the sample responds in this way. As the number of individuals grows, the severity of the problem will increase. It may thus be valuable to keep the sample size small in order to maintain the incentives for introspection or information acquisition, and accurate reporting of tastes.

This can be modelled as follows:

We first consider a typical agent who is one among $n + 1$ sampled individuals. It is assumed that the sample size is announced to the agents so that they know, with certainty, that there are n others. Let x denote the sum of the answers of these n others.

Let $B(v_i, w_i)$ be the agent's gain if he says v_i , the truth, instead of w_i . Clearly, $B(v_i, w_i)$ depends on x :

$$B(v_i, w_i) = 0 \quad \begin{array}{l} \text{if } v_i + x \geq 0 \text{ and } w_i + x \geq 0 \\ \text{or if } v_i + x < 0 \text{ and } w_i + x < 0 \end{array}$$

$$= v_i + x \quad \text{if } v_i + x \geq 0 \text{ and } w_i + x < 0$$

$$= -v_i + x \quad \text{if } v_i + x < 0 \text{ and } w_i + x \geq 0$$

We assume that the agent believes that the n others will make statements such that the total is distributed according to a probability law $p_n(x)$. (We will see that this distribution will itself be determined endogenously.) The expected gain of saying the truth, v_i , instead of saying w_i is thus

$$E_x B(v_i, w_i) = \int_{\substack{w_i \leq -x < v_i \\ -w_i \leq x < -v_i}} |v_i + x| dp_n(x)$$

A typical individual is assumed to believe that his own v_i , initially unknown to him, is distributed normally with mean zero and variance r^2 . At a cost c (in terms of the private good), he can learn v_i precisely. Moreover we assume that he knows that all other individuals are in exactly the same position, except for the fact that their discovery costs may be different from his. In any situation, he is therefore justified in assuming that those whose costs are low will ascertain their v_i and will correctly report them, whereas those with high costs will report zero. The cutoff value of costs will be denoted c^* ; it will be determined endogenously as a function of n .

For any c^* , the typical agent is assumed to know the proportion of the sample whose value of c is below c^* , $F(c^*)$, where $F(\cdot)$ is the distribution of costs in the population. We neglect the uncertainty that sampling will induce concerning the exact size of the group of agents whose values of c are below c^* . Hence the agent believes that $nF(c^*)$ individuals will respond according to their true v_i , drawn from a distribution $N(0, r^2)$ whereas $n(1 - F(c^*))$ will respond with zero. Thus his subjective distribution on x , $p_n(\cdot)$ is $N(0, nF(c^*)r^2)$.

Before ascertaining his own v_i , the expected gain from doing so is

$$E_{v_i} E_x B(v_i, 0) = E_{v_i} \int_{\substack{w_i \leq -x < v_i \\ -w_i \leq x < -v_i}} |v_i + x| dp_n(x)$$

where the expectation over v_i is taken with the distribution $N(0, r^2)$. Thus, balancing the expected gain from information acquisition against cost, the individual will choose to learn his v_i only if

$$E_{v_i} E_x B(v_i, 0) \geq c$$

Thus an equilibrium value of c^* is one that gives rise to an equality in this relation. It can be shown (see Green and Laffont, 1976a) that $c^*(n)$ must satisfy

$$\frac{r}{\sqrt{2\pi}} [\sqrt{nF(c^*) + 1} - \sqrt{nF(c^*)}] = c^*$$

and that the solution to this will be unique and decreasing in n . Thus as the planner chooses different values of n , he will obtain more accurate answers depending on the change in $nF(c^*(n))$.

As in section 6.2.1 we assume that the planner chooses the sample size to maximize the expected value of the outcome, net of costs, believing that the population is drawn from $N(\nu, \sigma^2)$ and that σ^2 is known but ν is unknown. His prior on ν is given by $N(0, p^2)$. In order to preserve the parity between the knowledge and beliefs of the planner and those of the individuals, it is natural to take $\sigma^2 + p^2 = r^2$. In this way the agents' beliefs about each other before ascertaining their own tastes are the same as the government's beliefs about each agent (before the sample's responses are known).

In a manner precisely analogous to that in section 6.2.1, we can therefore derive that the expected value of the public decision, ex ante is given by

$$\frac{(Np^2 + \sigma^2)}{(2\pi)^{1/2} \left(\frac{\sigma^2}{nF(c^*(n))} + p^2 \right)^{1/2}} \quad (6.6)$$

The costs to be subtracted from this are the mean costs of private information acquisition incurred by individuals below the cutoff level of c^* .

$$\int_0^{c^*(n)} c dF(c) \quad (6.7)$$

The optimal value of n is that which maximizes the difference between (6.5) and (6.7). Because of the analytical complexity of $c^*(n)$ it is, in general, impossible to obtain results on the optimal sample size. However, by making further assumptions on the distribution of the information costs, such results can be attained. If $F(c)$ is of the form

$$F(c) = \frac{c^m}{K_m}$$

where $m \in [0, 1]$ and K_m is a constant, then a closed-form solution for c^* is

$$c^* = \left[\frac{r}{2\sqrt{2\pi}} \right]^{2/2+m} K_m^{1/2+m} n^{-m/2+m}$$

and the optimal sample size, $n^*(N)$, can be shown to grow at the rate $N^{(2+m)/2m+2-m^2}$ when N is large. Specifically, this means that when $m = 1$, so that c is distributed uniformly on $[0, K_1]$, that $n^*(N)$ is proportional to N , in large populations.

One can then ask whether the induced private information costs are worth the social value of information produced. It can be shown that the optimized value of the sampling procedure approaches $p/\sqrt{2\pi}$ per capita. The process is not only informationally valuable in large economies, but the value increases linearly with N .

6.2.2.b. Preventing Coalitions from Distorting Their Responses. In the studies mentioned above, eliminating the possibility of rational distortion of preferences by individuals was the only goal of the system of transfer payments used. It can be easily seen, however, that if a coalition of individuals could form and, tacitly or explicitly, could agree to alter their responses in the same direction, a superior result from their point of view could be achieved. A simple example of this can be seen in the three person economy with $v_1 = -3$, $v_2 = +1$, $v_3 = +1$. Using any Groves mechanism each person has an incentive to respond with $w_i = v_i$. The project would be rejected but if individuals 2 and 3 could agree to each say +3, then the pivotal mechanism would lead to acceptance of the project with no transfers, and this is better than they would achieve by telling the truth. However, they cannot be assumed to know the statement that will be made by the other individual; or more generally they will even be ignorant of how many other individuals there are. Distortions will involve some risks. For example if $w_1 = -5$, then the distortion mentioned above would lead to both individuals being taxed 2 units by the pivotal mechanism, which is more than their true willingness to pay for the project they have caused to be accepted. The larger their distortion, the surer they are of forcing their preferred social decision, but the more risk they must accept if their guess about the others' statements proves erroneous.

In Green and Laffont (1976d) we show that there is no successful mechanism that is capable of inducing truthful responses from coalitions as well as individuals. A related result was demonstrated by Bennett and Conn (1976).

The natural question then becomes: Given that there is no way to prevent cheating by coalitions in all circumstances, can we design systems with the property that coalitions will not find it very profitable to form? In this way, if there are any costs of collusion, communication or enforcement within the coalition, or if there are any problems of elicitation of preferences within the coalition itself, these will form a sufficiently strong barrier to their formation.

The prospect of sampling is a possible advantageous device in this regard—though the matter is not one of a simple optimization of the sample size as in the sections above. There are conflicting forces determining the optimal sample size, and some are difficult to quantify. On one hand a small random sample disbursed among the members of a large population may find it exceedingly difficult even to seek each other out and identify their common interests. On the other hand, since it is the variability and uncertainty concerning the responses of others that causes the risk of collusion, we should take a larger sample in order that each coalition perceives itself as facing a larger group of others in the sampled group. If they believe the statements to be independently and identically distributed, as we have assumed in other sections, then they will be less likely to try to distort their preferences.

We have shown (Green and Laffont (1976d)) that as the sample size grows, the expected profitability from cheating by the coalition for whom such distortion is potentially the most valuable falls to zero. This is done in several steps. We first study the optimal response by a coalition of agents whose willingness to pay have the same sign and who have made a binding agreement to share their taxes, if any, in a particular way. It is shown that the best joint response is to have all coalition members say the same thing, and for this to be approximately equal, for each individual, to the willingness to pay of the entire coalition.

Next, fixing the aggregate willingness to pay of the members of the coalition, it is shown that the expected gain to cheating compared to telling the truth, goes to zero with the number of agents in the sample. However, if the planner shares the individuals' beliefs about the distribution of willingnesses to pay, then as the sample size increases the probability that a coalition of a fixed size exists with an aggregated willingness to pay exceeding a particular quantity increases. Nevertheless, we show, by using the asymptotic theory of order statistics, that if these beliefs are normally distributed, the probability that a viable cheating coalition will exist with an expected per capita gain larger than some fixed positive number will approach zero as the sample size grows.

Of course the risks of collusion and the costs of eliminating it would have to be balanced against the other problems involved in the employment of these mechanisms. These more complex comparisons remain open questions that have not been studied thus far.

6.3. Participation

We have seen that it may be superior, for a variety of reasons, to elicit the preferences of only a sample from the whole population. In a noncoercive society, however, we cannot insure that all the members of the sampled group

will respond by giving their true tastes. A reply of "no response" must be admitted as a valid individual strategy, in addition to the elements of the real line which form the allowable valuations for the project as we have posed the question thus far.

We are therefore faced with a choice of either designing mechanisms such that universal participation can be insured, or of specifying the outcomes that will arise in the event that the reply of "no response" is played by some of the agents. Because failures to respond may bias the outcome of the process, we have opted for the former possibility.

At this stage we must distinguish between alternative assumptions concerning participation. We might require, on one hand, that each individual should prefer to participate as a member of the sample instead of abstaining and leaving the decision making process to the others. This we term the requirement of *universal participation*. On the other hand, a more traditional approach is to require that no one can be hurt because of his participation—vis-à-vis the *status quo*. This is the requirement of *individual rationality*.

Due to the fact that no individually incentive compatible mechanism can balance the budget identically (see Green and Laffont 1976b) the members of the unsampled group will be paying or receiving transfers from the sampled group as a whole. This welfare will be further affected by the public decision taken. If a net subsidy to the sample is required the unsampled group is likely to contain at least some individuals who are made worse off by virtue of the use of this mechanism. We may ask, therefore, for mechanisms whose net transfer to the sample is always nonpositive, so that it will then be possible to treat the unsampled group in a welfare improving way, at least as far as monetary transfers are concerned. Indeed it is clear that it is impossible to design a mechanism which is individually rational for the whole population since an agent not in the sample may have an arbitrary (negative) evaluation.

We will say that a mechanism is *weakly individually rational* if it is individually rational for every member of the sample and generates a nonnegative transfer to the unsampled group.

Finally, the decision regarding participation, under either criterion, may depend on the individuals' beliefs regarding the statements made by other members of the sample. An interesting feature of the dominant strategy mechanisms in this regard is that the requirement of universal participation, independent of expectations, leads to a class of mechanisms defined by a set of functions $h_i(\cdot)$ which are bounded below by the pivotal mechanism—which we found useful in section 6.2 in several contexts. Thus, if the government wants to minimize the total subsidies required by these procedures (or maximize the revenues they collect) then the pivotal mechanism should be selected within the class inducing universal participation. The pivotal mechanism would therefore have a very strong claim on our attention.

6.3.1. Universal Participation

Theorem 6.3.1

Let

$$h_i(w_{-i}) = \min(-\sum_{j \neq i} w_j, 0)$$

define the pivotal mechanism. Let

$$h'_i(\bar{w}_{-i}) < h_i(\bar{w}_{-i}) \text{ for some } w_{-i}$$

Then there exists $v_i \in R$ and expectations concerning $\bar{w}_{-i}$ such that an individual with these tastes and beliefs would refuse to participate if the mechanism were defined by $h'_i(\cdot)$.

Proof

It suffices to consider $v_i = 0$, and expectations concentrated on $(\bar{w}_{-i})$. The expected payoff to such an individual will be

$$\begin{aligned} \sum_{j \neq i} w_{-j} + h'_i(\bar{w}_j) & \quad \text{if } \sum_{j \neq i} \bar{w}_j \geq 0 \\ h'_i(\bar{w}_{-i}) & \quad \text{if } \sum_{j \neq i} \bar{w}_j < 0 \end{aligned}$$

Since $h'_i(\bar{w}_{-i}) < h_i(\bar{w}_{-i})$, we will have a negative quantity in either case, from the definition of $h_i(\cdot)$. Hence the individual would be unambiguously worse off if he were to participate, as he would sustain a certain loss.

Thus any mechanism that gives lower transfers at any point than the pivotal mechanism will not necessarily induce universal participation.

We assume throughout this section that individuals are all risk-neutral. A mechanism induces participation by an individual, i , whose beliefs are given by $g(\cdot)$, a distribution on $\bar{w}_{-i}$, only if the expected value of his payoff is nonnegative.

Theorem 6.3.2

For any distribution $g(\cdot)$, an individual with $v_i = 0$ will choose to participate in the sampled group if and only if the mechanism would induce universal participation for any individual with these expectations.

Proof

The expected utility, given participation, can be written as:

$$\int_{\substack{j \neq i \\ \sum_{j \neq i} w_j = -v}}^{\sum_{j \neq i} w_j = \infty} (v + \sum_{j \neq i} w_j) dg(w_{-i}) + \int_{-\infty}^{+\infty} h_i(w_{-i}) dg(w_{-i})$$

Nonparticipation would induce the expected utility,

$$\int_{\substack{j \neq i \\ \sum_{j \neq i} w_j = 0}}^{\sum_{j \neq i} w_j = 0} v_i dg(w_{-i})$$

The difference between them is, therefore,

$$\int_{\substack{j \neq i \\ \sum_{j \neq i} w_j = -v}}^{\sum_{j \neq i} w_j = \infty} (v_i + \sum_{j \neq i} w_j) dg(w_{-i}) + \int_0^{+\infty} \sum_{j \neq i} w_j dg(w_{-i}) + \int_{-\infty}^{+\infty} h_i(w_{-i}) dg(w_{-i})$$

Note that the first integral is nonnegative, whereas the second two are independent of v_i . Further, as $v_i \rightarrow 0$, the first integral converges to zero.

Therefore, if for some v_i the mechanism does not induce participation, then it will not induce participation by an individual whose $v_i = 0$. Moreover, if the second two integrals are nonnegative, then an individual for whom $v_i = 0$ will participate, and hence so will all individuals.

Corollary:

The set of all mechanisms that induce universal participation by all risk-neutral individuals is characterized by the set of all $h_i(\cdot)$ such that

$$h_i(w_{-i}) \geq \min(-\sum w_{-i}, 0)$$

Proof

For given expectations $g(\cdot)$, universal participation requires

$$\int_0^{\infty} \sum_{j \neq i} w_j dg(w_{-i}) + \int_{-\infty}^{+\infty} h(w_{-i}) dg(w_{-i}) \geq 0$$

Since it must hold for any expectations $g(\cdot)$ it implies

$$h_i(w_{-i}) \geq \min(-\sum_{j \neq i} w_j, 0) \quad Q.E.D.$$

6.3.2. Weak Individual Rationality

We have defined individual rationality to mean a welfare improvement for all members of the population as a result of using the mechanism. When a sample is taken, it is clear that this requirement can never be satisfied with certainty for individuals not in the sample. Some individuals can have a large, negative evaluation for the project while the sample can accept it. Therefore the best that can be hoped for is to have the nonsampled individuals receive a positive monetary transfer, albeit perhaps not enough to compensate them fully. It will be shown, however that even this more modest goal, is unattainable, if individual rationality for members of the sample is to be simultaneously insured. We demonstrate this in the following two theorems, which characterize the class of h functions required for the sampled and unsampled groups. It will then be shown that these are inconsistent.

Theorem 6.3.3

If $h_i(\cdot)$ are nonnegative for all i , then the mechanism is individually rational for the sampled group, and conversely.

Proof

If the project is rejected, $h_i \geq 0$ insures welfare improvement. If the project is accepted, individual i 's change in welfare is

$$v_i + \sum_{j \neq i} w_j + h_i(w_{-i}) = \sum_j w_j + h_i(w_{-i})$$

Since $w_i = v_i$ by the dominant strategy property, and both terms are nonnegative.

If $h_i(\bar{w}_{-i}) < 0$ for some $\bar{w}_{-i}$, then letting

$$v_i = -\sum_{j \neq i} \bar{w}_j$$

the project will be accepted ($\sum_j w_j = 0$) and the change in utility will be

$$v_i + (\sum_{j \neq i} \bar{w}_j) + h_i(\bar{w}_{-i}) = h_i(\bar{w}_{-i}) < 0.$$

Theorem 6.3.4

If

$$t_k(w_{-k}) \geq \min(-\sum_{j \neq k} w_j, 0) \text{ for all } k$$

and

$$h_i(\bar{w}_{-i}) > \min(-\sum_{j \neq i} \bar{w}_j, 0) \text{ for some } i \text{ and some } \bar{w}_{-i}$$

then there exists a number, $\hat{w}_i$, such that the sampled group will receive a positive subsidy if the statements are $(\bar{w}_1, \dots, \bar{w}_{i-1}, \hat{w}_i, \bar{w}_{i+1}, \dots, \bar{w}_N)$.

Proof

There are two cases according to the sign of $\sum_{j \neq i} \bar{w}_j$. If $\sum_{j \neq i} \bar{w}_j \geq 0$, take $\hat{w}_i$ large and positive. Then the transfer to i will be

$$\sum_{j \neq i} \bar{w}_j + h_i(\bar{w}_{-i}) > 0$$

By taking $\hat{w}_i$ sufficiently large, no other individual will be pivotal and hence no one else will have a negative transfer. Thus all transfers are nonnegative and their sum is strictly positive. The case of $\sum_{j \neq i} \bar{w}_j < 0$ is handled analogously by taking $\hat{w}_i$ large and negative.

It may be, however, that we can have h functions above the pivotal one for some individuals while not for others and still maintain the nonnegativity of the transfer to the unsampled group in all situations.

However, if $h_i \geq 0$ is required for the individual rationality of the sampled group and $h_i \leq \min(-\sum_{j \neq i} \bar{w}_j, 0)$ is required for positive transfer to the unsampled group, there is a clear contradiction as the two are incompatible wherever $\sum_{j \neq i} \bar{w}_j > 0$. It is therefore impossible to maintain a weakly individually rational and incentive compatible mechanism under these definitions.

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