STOCHASTIC EQUILIBRIUM: A STABILITY THEOREM AND APPLICATION

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I. Introduction

This paper is a continuation of the study initiated in [5]. Our concern is with the definition of equilibrium in a model in which there is an inherent stochastic element in the behavior of the economic agents. This is to be distinguished from cases in which the environment of some agents is either stochastic or is viewed with subjective uncertainty but in which the solution to the individual optimizing problem is deterministic.

The analysis is set in a simple general equilibrium trading model in which agents have stochastic tastes. This model, while perhaps interesting in its own right, is not the only situation to which our equilibrium concept and theorems are applicable.

We proceed below as follows:

Since self-containment is a virtue, we next present and briefly discuss the definition of equilibrium in the stochastic choice setting. Readers familiar with [5] may skip this without loss. Next the reasons for interest in a stability result for this system are discussed. This discussion is relevant to the related works of Hildenbrand [6] and

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Majumdar and Bhattacharya [1,8]. Comparisons are made between the equilibrium concepts used by these authors and the one treated here, and the relationship of stability analysis to each.

Then the main theorem is presented and proven, followed by a demonstration of a set of conditions under which the necessary hypotheses would be fulfilled.

Lastly, we present discussions of how this equilibrium concept would apply to a situation other than the stochastic choice model. The resulting equilibrium is discussed and compared with the treatments received in the literature.

II. Recapitulation of the Equilibrium Concept Used and a Statement of the Problem at Hand.

In a situation in which agents demands depend on a random variable as well as price, and prices adjust if observed excess demand is not zero, no single price system could be an equilibrium.[1/] Excess demands given this price would be a random variable and the result of the price adjustment process is therefore random. The concept of equilibrium we shall use is that it is a distribution over the price space that is stationary with respect to the stochastic demand function and the adjustment mechanism. In [5] the existence of such a stationary distribution was demonstrated under very mild conditions.

We assume that there are k commodities and that there is an underlying probability space through which demand functions are generated, $(\Omega,\mathcal{B},\mu)$. $\mathcal{B}$ is taken to be the Borel σ-field of the topology on Ω. Let Δ be the price simplex. Excess demand functions are mappings ζ^i (for the i^{th} individual) from $\Omega\times$ int Δ to R^k. We assume that ζ^i is a continuous function of both arguments.[2/] Let ζ be the aggregate excess demand function. Let $h: R^k \times \text{int } \Delta \to \text{int } \Delta$ be the price adjustment mechanism. Assumptions are made to insure the existence of a compact subset of int Δ such that $h(\zeta(\omega,p),p)$ is an injection on this subset.

The main part of the proof is to show that the mapping from any measure to the one generated by it is continuous. A fixed point theorem is then applied to obtain the existence of a stationary measure.

We now ask, in what sense can such a stationary measure be justified as an equilibrium. The existence theorem merely suffices to show that if prices are initially chosen according to the stationary measure, then a priori we should expect the same measure to generate the prices in any future period. Yet this is in some sense unsatisfactory. Prices are not chosen randomly; no one could know exactly what measure to use to achieve stationarity even if they were drawn this way. Indeed all one can observe is particular price vectors that emerge -- the stationary distribution is at best a statistical phenomenon.

However, suppose one could show that for all initial prices, the sequence of measures generated converges to the stationary distribution weakly. Then this measure would take an added significance since it would represent the long-run distribution of prices irrespective of any initial conditions. This is the main result that we demonstrate below.

Of course global iterative stability (which implies uniqueness) is a very strong property. We shall have to make equally strong assumptions on the underlying demand functions and adjustment mechanism.

In the next section we state the assumptions of the main theorem and draw some parallels between these and one of the equilibirum concepts of Majumdar and Bhattacharya [1] and Hildenbrand [6].

III. Assumptions and Discussion

We change the notation slightly so that Δ (instead of Δ') is the compact set in the interior of the price simplex on which $h(\zeta(\omega,p),p)$ is an injection for μ-almost every $\omega \in \Omega$. To shorten this notation we write

$$g_\omega: \quad \Delta \to \Delta \quad \text{for each } \omega \in \Omega.$$

Assumption: g_ω is a contraction on Δ for each[3] $\omega \in \Omega$.

Because of the Banach fixed point theorem, this means that each deterministic economy, as defined by the sample ω has a unique globally stable equilibrium. The strength of this assumption in the general equilibrium setting hardly needs emphasis. However, in one of the applications of the main result, shown later in the paper, it is quite a bit less severe. We shall see that in certain partial equilibrium contexts the main theorem can be applied and the assumption above is equivalent to the absence of expanding or stationary cobweb phenomena.

Without such an assumption it is easy to see that the main theorem could not possibly be true. (Consider the trivial case in which Ω is a single point).

Consider one of the equilibria discussed by Majumdar and Bhattacharya [1] and Hildenbrand [6]. Each sample (deterministic) economy has an equilibrium p_ω and the equilibrium for the market with stochastic tastes is given by

$$\Pi^*(A) = \mu\{\omega \mid p_\omega \in A\} \text{ for all measurable subsets } A, \text{ of } \Delta.$$

The following story could be associated with such a description of the equilibrium state of the stochastic economy. Suppose the economy were repeated over time in such a way that the random elements were serially uncorrelated. In each period suppose that the (unique) equilibrium is globally stable and that the adjustment process (not explicit) is sufficiently swift so that equilibrium is attained and trading takes place within the period.[4/] Then Π^* represents the distribution of these observed equilibria.

In the case of the equilibrium concept we consider, the economy repeats in a temporally independant way each period; however prices adjust only one step of the way each time. Each period demand is based on the prices that result from the realization of the previous period, when the economy may have been different. The equilibrium distribution of prices is the long-run stochastic steady-state to which the economy will tend if each underlying deterministic economy would be globally stable _if_ it had sufficient time to adjust. One may thus think of this concept as an _equilibrium distribution_, in contrast with the Hildenbrand-Majumdar concept which is instead a _distribution of equilibria_.

While these concepts differ because they represent different answers to two distinct questions, there seems to be one reason to prefer the notion of this paper to the other.[5/] The issue is one of compatibility between our conceptualization of the timing of the adjustment process and the assumption of stationarity in endowment and stochastic demand functions. In the Hildenbrand-Majumdar definition, equilibrium is achieved in each period. If the adjustment takes a long time,[6/] the assumption that tastes next period are distributed identically with tastes this period becomes harder to swallow. So also does the stationary endowment assumption, particularly if we are concerned with the possibility of entry and exit of firms.[7/] The question of finite time convergence does not arise here since we take only one step of the adjustment process at each iteration of the stochastic process.

In this spirit it is easy to see that these concepts are simply the two extremes of a class of processes in which prices adjust T steps toward equilibrium between regenerations of the stochastic parameter. For the Majumdar-Hildenbrand equilibrium, set $T = \infty$; herein, set $T = 1$. In both cases global stability of the deterministic equilibria is a necessary condition.

In the next section we set forth the principle mathematical theorem and present the proof.

IV. Main Theorem and Proof

We shall let Δ be endowed with the metric $d(p,p') = \max_i |p_i - p_i'|$ and $\mathcal{F}$ be the Borel σ-field generated by the metric induced topology on Δ. M, the set of all probability measures on $(\Delta,\mathcal{F})$ is endowed with the weak topology. This is metrized by the Prohorov distance ρ, defined by

$$\rho(\Pi,\Pi') = \inf\{\in > 0 \mid \Pi(A) \leq \Pi'(A^{\in}) + \in \text{ and } \Pi'(A) \leq \Pi(A^{\in}) + \in \text{ for all } A \in \mathcal{F}\}$$

where $A^{\in} = \{p \in \Delta \mid d(p,A) < \in\}$.

We shall prove the following

<u>Theorem</u>: Let $(\Omega,\mathcal{B},\mu)$ be a probability space and let

$$g_\omega(\cdot): \Delta \to \Delta$$

be a contraction, for all $\omega \in \Omega$, with contraction constants $\delta_\omega \leq \delta < 1$.

Let f: $M \to M$ be defined by

$$f(\Pi)(A) = \int \mu\{\omega | g_\omega(p) \in A\}\Pi(dp)$$

Then f is iteratively globally stable. That is, for all $\Pi \in M$, $\lim_n f^{(n)}(\Pi)$ exists and is independant of Π.

The theorem is established in two principle steps. First we show that $\rho(f(\Pi), f(\Pi')) \leqq \rho(\Pi,\Pi')$. That is, f never expands the distance between two measures. However, it is possible that f keeps the distance constant (we shall give an example), and is therefore not itself a contraction. The Banach fixed point theorem does not apply directly.

However, in the second part we show that if $\rho(\Pi,\Pi') = D$, then there exists N such that $\rho(f^{(N)}(\Pi), f^{(N)}(\Pi')) \leqq D\delta$ (where δ is the bound on the contraction constants of the $g_\omega(\cdot)$). N may depend on D but not on the particular choice of Π and Π'. Lastly we show that these two results are sufficient to insure the stability of f in the above sense.

Lemma 1: $\rho(\Pi,\Pi') \leqq \rho(f(\Pi), f(\Pi'))$.

Proof of Lemma 1: The set of measures with support on finite set is dense in the weak topology. Thus, if we can show that for any Π,Π' with support on finite sets, $\rho(f(\Pi), f(\Pi')) \leqq \rho(\Pi,\Pi')$ we will have shown that this inequality is true for all $\Pi,\Pi' \in M$.

Thus for this part of the proof we shall let Π and Π' be measures concentrated on a finite set $\{a_1,\ldots,a_n\} \subset \Delta$, $\Pi(a_i) = x_i$, $\Pi'(a_i) = y_i$. (We may take both measures to have support on the same set by simply combining the supports and setting appropriate x_i's and y_i's equal to zero.) Let the Prohorov distance between Π and Π', $\rho(\Pi,\Pi')$ be less than or equal to $\in$. Thus for any subset A of $\{a_1,\ldots,a_n\}$,

$$\sum_{\{i|a_i\in A\}} x_i \leqq \sum_{\{i|a_i\in A^{\in}\}} y_i + \in$$

where $A^{\in}$ is defined by $\{a_j | d(a_i,a_j) < \in \text{ for some } a_i \in A\}$.
We must show that for any measurable subset $B \subseteq \Delta$, $f(\Pi)(B) \leqq f(\Pi')(B^{\in}) + \in$. Of course there is a second set of inequalities to show, but it will be obvious that this can be done by the other set of inequalities implied by $\rho(\Pi, \Pi') \leqq \in$. Since Π and Π' are concentrated on $\{a_1,\ldots,a_n\}$, it follows that

$$f(\Pi)(B) = \sum_{i=1}^{n} x_i \; \mu\{\omega | g_\omega(a_i) \in B\}$$

and

$$f(\Pi')(B^{\in}) = \sum_{i=1}^{n} y_i \; \mu\{\omega | g_\omega(a_i) \in B^{\in}\}.$$

Let $B \in \mathfrak{F}$ be arbitrary and fixed and let

$$\mu\{\omega | g_\omega(a_i) \in B\} = \mu_i$$

and

$$\mu\{\omega | g_\omega(a_i) \in B^{\in}\} = \mu_i^+$$

If $d(a_i,a_j) \leqq \in$ and $g_\omega(a_i) \in B$, then $g_\omega(a_j) \in B^{\in}$ since g_ω is a contraction (actually $g_\omega(a_j) \in B^{\delta\in}$ but we do not need this). Thus, if $d(a_i,a_j) \leqq \in$, $\mu_i^+ \geqq \mu_j$.
Let D be the binary relation on $\{1,\ldots,n\}$ defined by iDj if and only if $d(a_i,a_j) \leqq \in$. It is clear that D is reflexive and symmetric (but not necessarily transitive).
Thus the lemma that $\rho(f(\Pi),f(\Pi')) \leqq \rho(\Pi,\Pi')$ is equivalent to the following proposition:

Proposition: Let D be a symmetric reflexive binary relation on $\{1,\ldots,n\}$. Let $<x_i>$, $<y_i>$, $<\mu_i>$, $<\mu_i^+>$ be four vectors indexed over $\{1,\ldots,n\}$. Let

$$D(i) \equiv \{j \mid jDi\},\ i \in N$$

$$D(I) \equiv \{j \mid jDi, \text{ for some } i \in I\},\ I \subseteq \{1,\ldots,n\}.$$

Assume that $x_i, y_i \geqq 0$

$$0 \leqq \mu_i \leqq 1$$

$$0 \leqq \mu_i^+ \leqq 1 \quad \text{for all } i.$$

If

(1) $j \in D(i)$ implies $\mu_j^+ \geqq \mu_i$

(2) for all $J \subseteq \{1,\ldots,n\}$

$$\sum_J x_i \leqq \sum_{D(J)} y_i + \in$$

Then

$$\sum_{i=1}^n x_i\mu_i \leqq \sum_{i=1}^n y_i\mu_i^+ + \in$$

The proof of the proposition proceeds in two steps. We first establish the following

Lemma to the proposition: Assume the conditions of the proposition. Let $J_1 \supset J_2 \supset \ldots J_\ell$ be a collection of nested subsets of $\{1,\ldots,n\}$. Let $\mu = <\mu_1,\ldots,\mu_n> = \sum_k \alpha_k \chi_{J_k}$ where $\alpha_k \geqq 0$ and χ_{J_k} is the characteristic function of the set J_k. Then $\mu^+ = <\mu_1^+,\ldots,\mu_n^+> \geqq \sum_k \alpha_k \chi_{D(J_k)}$.

Proof[8/]: Consider $j \in \{1,\ldots,n\}$. If $j \notin D(J_k)$ for all k, then

$$\mu_j^+ \geqq 0 = \sum_k \alpha_k \chi_{D(J_k)}(j)$$

since $\chi_{D(J_k)}(j)=0$ for all k, so there is nothing to prove.

Suppose then that j is in $D(J_1),\ldots,D(J_m)$ but not in $D(J_{m+1})$. Thus there exists $i \in J_m$ such that $j \in D(i)$ and $i \in J_k$ for all $k \leqq m$. Hence

$$\mu_i = \sum_{k \leqq m} \alpha_k \chi_{J_k}(i) = \sum_{k \leqq m} \alpha_k$$

By condition (1) of the proposition

$$\mu_j^+ \geqq \mu_i \geqq \sum_{k \leqq m} \alpha_k = \sum_{k \leqq m} \alpha_k \chi_{D(J_k)}(j)$$

$$\geqq \sum_{k=1}^{\ell} \alpha_k \chi_{D(J_k)}(j), \text{ since } \chi_{D(J_k)}(j) = 0 \quad k > m.$$

This completes the proof of this lemma.

Proof of the proposition: We shall construct a collection of subsets $J_1 \supset \ldots \supset J_\ell$ and a collection $\{\alpha_k\}$ below, to which we shall apply the above lemma

Let

$$J_1 = \{i \mid \mu_i > 0\}$$

and $$\alpha_1 = \min_{i \in J_1} \mu_i .$$

Let $$J_2 = \{i \mid i \in J_1,\ \mu_i > \alpha_1\}$$

and $$\alpha_2 = (\min_{i \in J_2} \mu_i - \alpha_1).$$

Similarly, $$J_{k+1} = \{i \mid i \in J_k,\ \mu_i > \alpha_k\}.$$

$$\alpha_{k+1} = \min_{i \in J_{k+1}} \mu_i - \alpha_k .$$

It is clear that this process terminates since the sequence of sets is strictly decreasing by construction. Also, by construction,

$$\mu = \sum \alpha_k \chi_{J_k}$$

Note, $\sum\limits_k \alpha_k = \max\limits_i \mu_i \leqq 1$.

Applying the lemma we obtain,

$$\sum_{i=1}^{n} x_i \mu_i = \sum_{i=1}^{n} x_i \sum_k \alpha_k \chi_{J_k}(i)$$

$$= \sum_k \alpha_k \sum_{i=1}^{n} x_i \chi_{J_k}(i)$$

$$= \sum_k \alpha_k \sum_{i \in J_k} x_i$$

$$\leqq \sum_k \alpha_k \Big(\sum_{i \in D(J_k)} y_i + \epsilon \Big) \qquad \text{by 2}$$

$$= \sum_k \alpha_k \sum_{i=1}^{n} y_i \chi_{D(J_k)}(i) + \epsilon \sum_k \alpha_k$$

$$\leqq \sum_{i=1}^{n} y_i \sum_k \alpha_k \chi_{D(J_k)}(i) + \epsilon \sum_k \alpha_k .$$

$$= \sum_{i=1}^{n} y_i \mu_i^+ + \epsilon$$

since $\mu_i^+ \geqq \sum\limits_k \alpha_k \chi_{D(J_k)}(i)$ by the lemma, and $\sum \alpha_k \leqq 1$ by construction. QED

This completes the proof of lemma 1. However, the following example shows that we may not strengthen this result further to conclude that f is a contraction, but rather only that f does not increase Prohorov distance.

Example: Let Π^1 and Π^2 be concentrated on a_1 and a_2 with weights $(x, 1-x)$ and $(y, 1-y)$ respectively, (with $x \neq y$).

Let $d(a_1, a_2) = 2|x-y|$.

Then $\rho(\Pi^1, \Pi^2) = |x-y|$.

Let Ω consist of a single point and let g_ω have contraction constant .9.

Then

$$d(g_\omega(a^1),\ g_\omega(a^2))$$

may be as large as $1.8\ |x-y|$ in which case

$$\rho(f(\Pi^1),\ f(\Pi^2)) = |x-y|,$$

and hence the inequality is only weak.

Lemma 2: Let Π^1 and Π^2 be two measures concentrated on $\{a_1,\ldots,a_n\}$. Let $D_A = \inf\ \{\in\ > 0 | \Pi^1(A) \leqq \Pi^2(A^\in) + \in\}$ where $A \subseteq \{a_1,\ldots,a_n\}$, and $D = \max\limits_A D_A$.

Then there exists N_D independent of Π^1, Π^2, but dependent on D, such that

$$f^{(N)}\ (\Pi^1)(B) \leqq f^{(N)}(\Pi^2)(B^{D\delta}) + D\delta$$

for all $B \in \mathcal{F}$. Further, taking N_D such that $\delta^{N-1} < D$ suffices.

Thus even though f is not a contraction, $f^{(N_D)}$ contracts any two measures with distance D between them (with the same contraction constant as the bound on the contraction constants of g_ω). We are motivated by the above example in which it may be seen that for N large enough that $(.9)^{N-1} < |x-y|$, we have that

$$d(g_\omega^{(N)}(a^1),\ g_\omega^{(N)}(a^2)) < (.9)\ |x-y|$$

and hence that

$$\rho(f^{(N)}(\Pi^1),\ f^{(N)}(\Pi^2)) < .9\ |x-y|.$$

Of course the situation is more complex when Ω has more than one point, but the basic idea of the proof of the lemma below follows along these lines.

Proof of lemma 2: Choose N such that $\delta^{N-1} < D$, where δ is the bound on the contraction constants of the g_ω. Let μ^N be the natural measure induced by μ on the N-fold cartesian product of Ω, Ω^N. That is if Q is a measurable rectangle in the product σ-field, $\mu^N(Q) = \mu(Q_1) \times \mu(Q_2) \times \ldots \times \mu(Q_N)$ where Q_i is the projection of Q into the i^{th} factor space, Ω. Fix $B \in \mathcal{F}$.

Let $a_i^N = \{p \mid p = g_{\omega_N}(g_{\omega_{N-1}}(\ldots(g_{\omega_1}(a_1))\ldots))$ for some $<\omega_1,\ldots,\omega_N> \in \Omega^N\}$

Let $B_i = B \cap a_i^N$

Let $O_i = \{<\omega_1,\ldots,\omega_N> \mid g_{\omega_N}(g_{\omega_{N-1}}(\ldots(g_{\omega_1}(a_i))\ldots)) \in B_i\}$

Let $B_{ij} = \{p \mid p = g_{\omega_N}(g_{\omega_{N-1}}(\ldots(g_{\omega_1}(a_j))\ldots))$ for some $<\omega_1,\ldots,\omega_N> \in O_i\}$.

In words, a_i^N is the set of all points that could be reached in N iterations of g from a_i; B_i are the parts of these sets in the set in question, B -- the B_i can intersect. O_i is the set of samples in Ω^N that would lead a_i to B. B_{ij} is the set of prices that would be reached from a_j if a sample occurred that would have brought a_i into B if we had begun there instead.

From the above definitions, the definition of N, and the fact that each of the g_ω is a contraction with constant less than δ, one can see that

$$\bigcup_j B_{ij} \subseteq B_i^{D\delta}.$$

This is true because the maximum distance possible between a_i and a_j is 1 since they are both in Δ which is the unit simplex with supremum norm. Thus, beginning from a_j after N iterations, one is necessarily within δ^N of where one would be having begun from a_i and drawing the same sample. The above inclusion expresses this statement for samples which would take a_i to B. Taking unions over i,

$$\bigcup_i \bigcup_j B_{ij} \subseteq \bigcup_i B_i^{D\delta} \subseteq B^{D\delta}.$$

Let $O = \bigcup_i O_i$. Claim:

$$f^{(N)}(\Pi^2)(\bigcup_i \bigcup_j B_{ij}) \geqq \mu^N(O).$$

Let a_k be arbitrary in $\{a_1,\dots,a_n\}$ and let $<\bar{\omega}_1,\dots,\bar{\omega}_n> \in O_m$ for some m. Thus $g_{\bar{\omega}_N}(\dots(g_{\bar{\omega}_1}(a_k))\dots) \in B_{mk}$ by definition.
Hence every sample in O leads to some $p \in \bigcup_i \bigcup_i B_{ij}$ irrespective of the initial point in the support of Π^2. This is sufficient to establish the claim. Let Π^1 have weights $x_1,\dots,x_n$ on $\{a_1,\dots,a_n\}$.
Now observe

$$\begin{aligned} f^{(N)}(\Pi^2)(B^{D\delta}) + D\delta &\geqq f^{(N)}(\Pi^2)(B^{D\delta}) \\ &\geqq \mu^N(O) \\ &\geqq \sum_i x_i\ \mu^N(O_i) \end{aligned}$$

since $\sum x_i = 1$, $x_i \geqq 0$ and $O_i \subset O$

$$\geqq f^{(N)}(\Pi_1)(B)$$

since $\mu^N(O_i)$ is the probability that a_i goes to B and x_i is the weight assigned by Π^1 to a_i. QED

The proof is now completed as follows: We show that given $\in > 0$, there exists $S_\in$ independent of Π and Π' such that $\rho(f^{(S_\in)}(\Pi), f^{(S_\in)}(\Pi')) < \in$. If Π is chosen arbitrarily and k is an arbitrary positive integer, let $\Pi' = f^{(k)}(\Pi)$. Then applying this result to Π and Π' yields

$$\rho(f^{S_\in}(\Pi), f^{S_\in+k}(\Pi)) < \in.$$

Thus the sequence $\Pi, f(\Pi), \ldots, f^{(n)}(\Pi)$ is Chauchy for all Π. Since the space M is complete, the limit exists. To see that the limit is independent of Π, simply apply lemma 2 to two such distinct sequences $f^{(n)}(\Pi_1) \to \Pi_1^*$ and $f^{(n)}(\Pi_2) \to \Pi_2^*$, as follows.

Let D be the distance from Π_1^* to Π_2^*. Choose $\bar{n}$ sufficiently large that $\rho(f^{(\bar{n})}(\Pi_1), f^{(\bar{n})}(\Pi_2)) \leqq \frac{D}{\delta}$. Then by lemma 2 there exists n* such that $\rho(f^{(n^*)}(\Pi_1), f^{(n^*)}(\Pi_2)) < D$. But since distances never expand so this is impossible since the limiting distance is D.

Thus we have shown that the following lemma suffices to prove the iterative stability of f.

<u>Lemma 3</u>: Given $\in > 0$ there exists $S_\in$ such that for all Π, Π', $\rho(f^{(S_\in)}(\Pi), f^{(S_\in)}(\Pi')) < \in$

<u>Proof of lemma 3</u>: By lemma 2, there exists N_D such that if $\rho(\Pi,\Pi') = D$, then $(f^{N_D}(\Pi), f^{N_D}(\Pi')) \leqq D\delta$. Further, if we let N_D be the smallest integer such that $\delta^{N_D-1} < D$, then the above conclusion is valid. Thus, when chosen in this way, $\in' > \in$ implies $N_\in > N_{\in'}$. Let T be an integer such that $\delta^T < \in$. Set $S_\in = TN_\in$. We shall prove that this suffices. Let $\rho(\Pi,\Pi') = D_0 \leqq 1$ since the maximum Prohorov distance between any two probability measures is 1. If $D_0 < \in$ then by

lemma 1 any $S_\in$ suffices so we are trivially done. If not, then by lemma 2, $\rho(f^{N_\in}(\Pi), f^{N_\in}(\Pi')) \leqq \delta D_0$. Again if $\delta D_0 < \in$ we are done. If not continue setting $(f^{kN_\in}(\Pi), f^{kN_\in}(\Pi')) = D_k$. After T iterations of this procedure $\rho(f^{(TN_\in)}(\Pi), f^{(TN_\in)}(\Pi')) \leqq \delta^T D_0 \leqq \delta^T < \in$. Q.E.D.

V. Discussion and Conclusion.

In the above sections it was shown that, in the stochastic choice setting, if each sample economy gives rise to a contraction $g_\omega(\cdot)$, then the stochastic equilibrium is globally stable. In this section we shall try to show that this result is applicable to a partial equilibrium situation -- one in which there are many sellers of a commodity each of whom makes their own pricing decisions, and many buyers each with imperfect knowledge about these prices.

To begin, we must ensure that the assumptions made in the above theorem are satisfied in this case. In particular, one should note that Walras' law plays only an indirect role in the main theorem. It's only use is to ensure the existence of an equilibrium in each sample economy ω. However, having assumed that $g_\omega(\cdot)$ is a contraction, the existence of an equilibirum for the sample economy is automatic, and independent of the proof in the general case, which depends on Walras' law. This gives us some hope for a partial equilibrium application of this result, since in such a situation Walras' law is not meaningful.

We envision the following situation. There are many sellers and many buyers, but more of the latter. Buyers sample the prices currently being quoted by sellers each period. There are so many sellers that, although sampling is costless (i.e., expends no real resources), time is such that it is possible to sample only a small proportion of the total number of sellers each period. Since every buyer knows that sellers change their prices each period, they do not keep track of who charged what in the past, for purposes of establishing price reputations to guide sampling in the future. Thus the first element of randomness is the set samples actually used by the different buyers. We assume that for a given set of samples, the buyer places orders at the various stores dependent on the prices observed. We do not assume that he necessarily attempts to buy at only the low priced stores. The reason for this is that he knows that the low priced stores are more likely to be oversubscribed (i.e., have orders in excess of output) than high-priced stores and hence may prefer to pay a higher price to increase the probability that the order is filled. An unfilled order has no force in following periods. The following description of a situation is one which suits the above conditions.

"Stores" are fish sellers each with a fixed number of boats and a fixed set of labor contracts with fisherman to operate them. The catch each day is (perhaps) a random variable and is distributed identically and independently each period. Fish are non-durable, if they are not sold they must be discarded. The price of fish is quoted in man-hours of labor used for boat repair. Each morning the boats go out, prices are set (we discuss below our assumption on the process that generates prices) and buyers pledge various amounts of their labor for boat maintenance in exchange for fish when the boats return that evening. If it turns out that there are not enough fish to satisfy all the contracts offered, some people go hungry (but do not do any work either). Further, the analysis is assumed to be sufficiently short-run so that strategies for offering repair labor are fixed.[9/] The important point here is that both fish and labor services are non-durable and no intertemporal contracts are permitted.

Sellers know that they are in a world of random excess demand. However, they are also ignorant of the prices of their rivals and perform sampling to better estimate the parameters of the demand distribution. For example, if a firm faces demand that it cannot satisfy, it would like to know whether this is due to the fact that its price is low compared to other sellers, or whether it is simply that it was lucky -- e.g., that it happened by chance that their price was sampled by an unusually large percentage of the people. However, the firm does not accumulate statistics on the stochastic demand function. As for buyers, we shall assume that no learning takes place, and that the policy that describes the price charged next period is fixed and is dependent on the excess demand faced by the firm and the sample drawn from the population of other prices. Thus we are not assuming that there is an equilibrium in expectations or in pricing policies.[10/] Neither are we assuming that the buying policies or pricing policies are optimal in any sense. All that is required is that they are stationary throughout the time frame of this analysis -- and since we envision this as a sequence of very-short-run situations (i.e., labor contracts for fishing and capital stock both fixed) this assumption does not seem inconsistent with the nature of the problem.[11/]

Thus, in our story about the Marshallian fish market there are several random elements -- the buyers sample, the sellers sample and (perhaps) the realization of the mixed strategies of the buyers and/or sellers. All of these are summarized by points in the space $(\Omega, \mathcal{B}, \mu)$. We make no assumption on their mutual dependence or independence. We hypothesize that for every point in this space there is a function $g_\omega(p)$ that gives the resulting set of prices if prices are originally p.

We now ask, under what conditions will $g_\omega(p)$ be a contraction in this framework; and under what conditions does there exist a compact subset of the price space such that $g_\omega(\cdot)$ is mapping from this set into itself for (almost) all $\omega \in \Omega$? If we can find conditions under which both of these are satisfied, the main theorem will apply directly. Thus we now discuss such a set of assumptions and conclude the paper by discussing the resulting equilibrium for this example.

We shall suppose that there are so many buyers relative to sellers that no matter what the sampling pattern is, there is a certain minimum number of buyers that will sample the price of any store.[12/] Further, due to capacity constraints on the amount of information any store can give out (perhaps one must visit the store to find out its price and it is possible for only a certain number of people to visit in one "day"), there is a maximum number of people who visit. Thus for any sampling pattern and any store, i, we suppose that there is a price

p_{min_i} such that excess demand is positive. That is, even if the minimal number of people come to the store, they will still demand more than the supply of that store. (Assume that the supply is non-random. generalization of this story to the case of random supplies is immediate). Similarly there exists p_{max_i} such that excess demand is surely negative. Then suppose that no matter what sample of prices in the rectangle generated by p_{min_i} and p_{max_i} is drawn by firms, or how much excess demand they face, the price of the i^{th} firm never changes by more than δ_i in one period and always changes in the direction of excess demand observed. Further assume, $\delta_i < p_{min_i}$. Thus the set $X = \{p | p_{min_i} - \delta_i \leq p_i \leq p_{max_i} + \delta_i\}$ is a compact set on which the price change function is an injection for every sampling pattern. Let the price adjustment function be $g(p)$, suppressing ω, and write

$$g(p) = h(\zeta(p), p).$$

Note that ω, in the situation described, would enter directly into the h function as well as ζ. However, this has no effect on our analysis. We take the sup metric on X, and assume differentiability as needed. Then the change in the k^{th} price is

$$dg^k(p) = \sum_i h^k_{\zeta_i} \sum_j \frac{\partial \zeta_i}{\partial p_j} dp_j + \sum_j h^k_{p_j} \; dp_j$$

Since each store can obtain information only about the prices changed by his competitors and not about the excess demands they face, we can write,

$$d\, g^k(p) = \sum_{j \neq k} [h^k_{\zeta_j} \frac{\partial \zeta_k}{\partial p_j} + h^k_{p_j}] dp_j + [h^k_{\zeta_k} \frac{\partial \zeta_k}{\partial p_k} + h^k_{p_k}] \, dp_k$$

It is not unreasonable to assume $h^k_{p_k} \in [0,1]$, $h^k_{\zeta_k} > 0$, $\frac{\partial \zeta_k}{\partial p_k} \leq \bar{\zeta} < 0$. Thus if the cross derivatives of h^k are sufficiently small, $|dg^k(p)| \leqq \alpha \max_j |dp_j|$ for all k, where $\alpha < 1$ is a constant dependent on $\bar{\zeta}$ and the terms on the first summation. Thus, while it requires rather severe assumptions to make g a contraction, at least it is plausible -- and it seems that it is not as restrictive as the global gross substitute condition of the deterministic stability theory. (Of course, that condition applies to continuous time adjustment processes, not discrete time as here[13/].) Another point worth mentioning is that responsiveness of h plays a crucial role in determining the stability of the process. This seems quite natural; but in the deterministic continuous time theory stability is often decided independent of the speed of adjustment.

Lastly we discuss what such a stochastic equilibrium might look like, and attempt to relate this to real world phenomena. If two stores are charging different prices for the same commodity, then in most samples, the low priced store will probably be facing a higher demand. Of course it is possible that the high priced store could get lucky and be sampled by more people, or that the realization of buyers mixed strategies makes the demand higher there. However, excess depends on endowments as well. If the low priced store is the one with more endowment, it is possible that excess demand is actually lower there on balance.

Looking at this the other way around, if two stores are of different size, it is likely that the stochastic equilibrium will have a distribution of prices that is usually lower at the larger store, but not necessarily always so. This is, in fact very commonly observed in the real world. Large (discount) (chain) stores charging lower prices than small (specialty) (local) ones.[14] This phenomenon of unequal prices (price distribution) in equilibrium is a result of the imperfect information on both sides of the market.[15]

In a context in which firms can change their size over time, it is not clear that they would all tend to the same size in the long run. It is quite possible that smaller, less efficient firms would have higher average profit margins in the stochastic equilibrium so that rates of return to entrepreneurs of all enterprises could be equal. Thus, the theory of stochastic equilibria might be used to explain the size distribution of firms, in some cases.

Footnotes

1. However, Hildenbrand [6] shows that if the choices are statistically independent, a single price will come close to equilibrating a very large market with high probability. Here we drop the independence assumption.

2. Majumdar has shown in [7] that this assumption could be weakened to measurable in $\omega \in \Omega$. In this case, no topological structure on Ω would be necessary and could be any σ-field of events

3. One could of course assume this only μ-almost everywhere in ω, but the resulting excess verbiage is not worth the increased generality. We systematically neglect sets of measure zero below, when it makes no difference.

4. Goods are non-durable and endowments are regenerated each period so that there is no real link between one economy and the next.

5. This is not meant to imply in any way that there is anything wrong with it.

6. Usually it is only proven that it is asymptotic, but we suppose for this discussion that equilibrium can actually be attained in finite time.

7. See Brock [3], Bohm [2].

8. I am indebted to Professor R. Aumann for the proof below.

9. Strategies need not be deterministic. In particular if we interpret them as being equilibria of a game between the buyers, they probably will be mixed.

10. See e.g. Brock [3].

11. In our interpretation of the Hildenbrand-Majumdar-Bhattacharya equilibrium (and I stress that this may not be that of any or all of these authors), this is less plausible. More time elapes between iterations of the stochastic process and it is thus less likely that the underlying process is stationary.

12. cf. Rothschild [9].

13. Also recall that the above discussion is in a partial equilibrium setting.

14. cf. Fisher [4] in which such a situation is called quasi-equilibrium, in contrast to full equilibrium in which all stores charge the same price for the same commodity.

15. cf. Rothschild [10].

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