

Compensatory transfers in two-player decision problems

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Abstract. This paper presents an axiomatic characterization of a family of solutions to two-player quasi-linear social choice problems. In these problems the players select a single action from a set available to them. They may also transfer money between themselves.

The solutions form a one-parameter family, where the parameter is a non-negative number, t .

The solutions can be interpreted as follows: Any efficient action can be selected. Based on this action, compute for each player a “best claim for compensation”. A claim for compensation is the difference between the value of an alternative action and the selected efficient action, minus a penalty proportional to the extent to which the alternative action is inefficient. The coefficient of proportionality of this penalty is t . The best claim for compensation for a player is the maximum of this computed claim over all possible alternative actions. The solution, at the parameter value t , is to implement the chosen efficient action and make a monetary transfer equal to the average of these two best claims. The characterization relies on three main axioms. The paper presents and justifies these axioms and compares them to related conditions used in other bargaining contexts. In Nash Bargaining Theory, the axioms analogous to these three are in conflict with each other. In contrast, in the quasi-linear social choice setting of this paper, all three conditions can be satisfied simultaneously.

Key words: bargaining, quasi-linear solutions, monetary transfers, cost allocation.

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1. Introduction

1.1. The problem

This paper concerns the normative analysis of two-player quasi-linear social choice problems. In these problems, two players select a single action from a set available to them. In addition, the players may make a payment of money to one another. Preferences are quasi-linear in this monetary transfer – relative evaluations of the outcomes are independent of the amount of money paid or received.

1.2. Solutions

A solution is a function that determines the action to be implemented and the payment to be made in each quasi-linear social choice problem. We seek solutions that always result in efficient outcomes and that have additional normative properties. In quasi-linear social choice problems, efficiency requires that the chosen action maximizes the sum of the players' evaluations; finding such actions is a straightforward calculation. The more subtle question therefore concerns how the monetary payments should vary with the problem that is faced.

We present an axiomatic structure that characterizes a one-parameter family of solutions. The parameter, denoted t , can take any non-negative value. The solutions and the interpretation of t are easy to describe:

Select any efficient action¹ and implement it. Then consider each player in turn. For the first player, and for each possible action, make the following computation: Take the difference between his evaluation of this action and the implemented action. Subtract t times the amount by which this action is inefficient – that is the quantity of money which, if added to the combined payoff of the two players, would make this action as efficient as the one that has been selected. Maximize this difference over all the actions². Think of the resulting quantity as the first player's proposal for a transfer of money in his favor. Now make the same calculation for the second player; think of the result as the second player's proposal. The solution, at the parameter value t , is to implement the selected efficient action, and to make a monetary transfer equal to the average of these two proposals.

1.3. Interpretation of solutions and the parameter t

The procedure outlined above gives a method for calculating the result of any solution obeying the axioms we propose. It describes this result as if there were a procedure for making and adjudicating claims for compensation. Yet the approach of this paper is entirely normative. The axioms describe desirable qualitative properties of solutions and do not mention "claims", "compensation" or "proposals" in any way. These terms help us understand

¹ If there is more than one, any can be selected. The solution, in the utility space, will be invariant to this selection.

² The maximal difference must be non-negative because if the action in question were equal to the implemented action, both terms would be zero.

the solutions and elucidate their behavior as the problem varies but are not part of the theory itself.

Monetary payments can be interpreted as compensation paid by or player to another for the latter's having forgone the opportunity to choose a different decision – one that this latter player would have preferred but which is inefficient for the group as a whole. For this reason we call the monetary payment a "compensatory transfer". In general, as can be seen from the calculation described above, both players may have a justifiable claim for compensatory transfer.

The parameter t represents a quantitative measure of the influence of "inefficient forgone alternatives" on the recommended result. A high value of t means that the solution tends to be less sensitive to such alternatives, and transfers will tend to be small in absolute value. A low value of t means that player will be well-compensated when his favorite alternative is not selected. At $t = 0$ the transfer will depend only on the maximal evaluation that each of the two players gives to any action.³

From the nature of the calculation method described, another property of solutions can easily be seen. The solution will depend only on the single "best proposal" that each player has available. The addition, deletion, or modification of actions that do not affect these two "best proposals" or the efficient selected action will necessarily leave the recommended outcome unchanged.

1.4. Applications

Quasi-linear social choice problems have a wide range of application. Cost allocation is one important area where they have been used⁴. In cost allocation problems, the payoffs are usually thought of as negative – efficient actions are those that result in the least negative aggregate payoff.

Many collective decisions in multi-division businesses or in multi-jurisdiction governmental settings fit naturally into the framework of this paper. The players are divisions, localities, or administrative units. One level higher in the organization than these "players" is a central authority that would like the players to take efficient decisions and to allocate the costs and benefits of these decisions equitably and consistently across problems. A related application is the allocation of corporate profits to divisions for reporting purposes.

We are also interested in using this model to evaluate the behavior of actual pairs of bargainers and of individuals who are called upon to make ethical judgments concerning problems faced by others. One of the advantages of having described a one-parameter family of solutions is that we can use experimental data to test the model⁵. If the model predicts well, we can

³ As discussed below in sections 1.6.4 and 6.1, the solution at $t = 0$ does not survive a natural strengthening of our continuity axiom. It is the only member of the one-parameter family characterized in Theorem 1 that fails this test. Thus, in some sense, this solution is a limiting case of the solutions with $t > 0$ which are to be preferred on theoretical grounds. See Theorem 2.

⁴ See Moulin [22] and Young [52], for a full discussion. Also Moulin [27] in Arrow, Sen and Suzumura [1], Champsaur [5], Kaneko [17], Loehmann and Whinston [18], and Chun [6][7][8].

⁵ See, for example, Bar-Hillel and Yaari [4].

then estimate the parameter value that is seemingly being used by pairs of bargainers or individuals whose behavior conforms to the theory.⁶

1.5. Axiomatic structure

In addition to the standard postulates such as symmetry, three other axioms are used to characterize a family of solutions. The first is Additivity: Solving independent problems should produce the same outcome whether they are approached separately or jointly. The second is a form of Monotonicity: Consider an action that is better for one of the players than any of the efficient actions. The existence of such an action creates an argument for a compensatory transfer in favor of this player. This argument should become stronger the less inefficient is the action.

The third axiom is called Recursive Invariance.⁷ Recursive Invariance contemplates a situation in which the solution is recommending that a transfer be made in order to reach a utility allocation that is not directly feasible by the choice of an action.⁸ In such a situation the axiom states that the addition of a new feasible action that happens to produce the solution's recommended utility allocation, without the need for a monetary transfer, should have no effect: the same utility allocation should be implemented. It could be reached either by retaining the original action-transfer pair, or by adopting the new action and making no transfers.

After formally presenting the axioms, we will argue that a failure of this invariance property would render the solution "vulnerable to renegotiation".

Recursive Invariance is not directly comparable to any axioms used in other approaches to bargaining theory although it is similar in spirit to "consistency" and "independence" axioms.⁹

1.6. Relationship to prior work

1.6.1. Normative bargaining theories

The normative bargaining literature falls into three groups. First, there is the Nash [29] approach, in which a feasible set of utility allocations and a status quo (or disagreement point) comprise the data of the problem. Second, there

⁶ Experimental work along these lines has been undertaken by the author, in collaboration with Dolly Chugh and Lorraine Idson.

⁷ This axioms has only been stated once before in the literature. See Chun [7]. Chun's term for this axiom is Trivial Independence. He shows that, in combination with Pareto Efficiency, it is implied by a variety of other axioms. Chun did not offer any separate justification for Trivial Independence. His paper was concerned with monotonicity and other comparative static properties of solutions, not with additivity. Trivial Independence was shown to be a logically weaker assumption than other postulates that were needed for his main theorem and was thus not separately explored.

⁸ Or a mixture of feasible actions.

⁹ Classic references on consistency and related ideas include Aumann and Maschler [3], Moulin

is the approach of games in characteristic function form¹⁰, in which the data of the problem are the total utilities that each subset of the players can achieve on its own. Third, there is the quasi-linear social choice framework.¹¹

The Nash framework allows the feasible set of utility allocations to be an arbitrary convex set. This allows for risk-averse players. In contrast, in the quasi-linear framework and in games in characteristic function form, risk-neutrality with respect to monetary transfers is assumed.

The disagreement point plays a special role in the Nash framework¹². In games in characteristic function form, the utility outcome where each player obtains the payoff he would receive if he were acting alone frequently plays the role of the disagreement payoff. In contrast, in the quasi-linear social choice model, if there were a status quo ante representing the situation before bargaining begins, implementing that point and making no transfer of money is treated on a par with all the other possible outcomes. It is not regarded as a "threat point", or given any other special role.

Quasi-linear social choice problems embody a two-level treatment of the utility possibility set. Some of the allocations result directly from the choice of an action, with no transfers being made. Other allocations can be reached only with non-zero transfers. The utility possibility set is a half-space – the set of all utility allocations whose sum is less than or equal to the maximal amount that can be achieved. The solution, lying on the boundary of this half-space, depends on the set of utility allocations that are reachable without the use of transfers. In this sense, the utility allocations that are induced directly by the choice of an action enjoy a different status, and have more salience within the theory, than those that are reached by the superposition of a monetary transfer upon a real decision. This distinction between the way in which feasible utility allocations are attained is not used in either the Nash-type theories or in games in characteristic function form.

While the two-level treatment of the feasible set might seem contrary to the utilitarian tradition, it is an essential part of the formulation of the problem we have posed. Consider the simplest possible problems in our domain – those in which there is only one possible action. A solution to these problems should not impose any monetary transfer because the role of transfers is to create equitable compromises based on the relationships between the action that is actually chosen and other actions that could have been chosen. However, in these simple problems the utility-possibility set is a half-space, as always. Only in the presence of inefficient foregone actions is there a need for transfers, and hence the theoretical necessity for our two-level treatment of the utility-possibility set.

The use of distributional information is another way in which the analysis of games in characteristic function form differs from that considered here.

¹⁰ We will restrict attention here to the transferable utility case. In this case, there is the implicit assumption of quasi-linearity of utilities – although the underlying set of actions and their evaluations are not usually mentioned in the analysis. See, for example, Aumann [2], and Roth [35]. For a discussion of games in characteristic function form without transferable utilities see Hart [13], or the survey Hart [14].

¹¹ These three approaches operate directly in the utility space. There is a fourth family of models as well – those that use information on how a utility allocation is achieved, as well as on what the utilities are. See Roemer [33] for an extensive discussion.

¹² See Peters [31] and Schmitz [37].

Each coalition is summarized by a single number, representing the best that group can do acting on its own. What is not stated or used in the analysis is what the distribution of this maximal payoff among the members of the coalition would be.

To summarize: Games in characteristic function form use information about payoffs for independent subgroups, whereas our analysis uses only the possibilities for both players acting in concert. Our theory neglects "threat points" or other results obtainable by individuals acting alone.¹³ At the same time, we do use information about the individual player's payoffs that would arise at all the decisions that could be made – both the inefficient and the efficient actions – whereas this distributional information is irrelevant to the analysis of games in characteristic function form.

1.6.2. Comparison of axiomatic structures and results obtained

The axioms in this paper are related to axioms that have been used elsewhere in the normative bargaining literature, either in the Nash or characteristic function frameworks, or in quasi-linear social choice theory. A full discussion will be given below, after the axioms have been presented formally. At this point, we will only highlight the differences in the results obtained.

The major difference between the results in this paper and those of other models is that in the present context the axioms are simultaneously satisfiable. We maintain the standard axioms of efficiency and continuity, and the widely used condition that the solution be independent of alternative utility representations. Additivity in the Nash context is generally inconsistent with efficiency, but an axiom in the same spirit, superadditivity, can be used to characterize a very interesting solution, see Maschler and Perles [19]. Monotonicity is used to characterize the Kalai-Smorodinsky solution, Kalai and Smorodinsky [16]. Of course the meaning of "superadditivity" and "monotonicity" is different in the Nash context, as the domain is different from that herein. Nevertheless, the fact that the Maschler-Perles and Kalai-Smorodinsky solutions are not the same demonstrates the irreconcilable tension between the ideas of additivity and monotonicity in the Nash context.

Concerning the independence of the axioms we note the following. In the presence of the standard axioms there are many solutions that satisfy Recursive Invariance but not Additivity or Monotonicity, and there are others that satisfy Additivity and Monotonicity but not Recursive Invariance. Nevertheless, the independence of the axioms from each other is not completely settled at this time. The proof of Theorem 1 entails the use of Monotonicity for technical reasons, yet we do not have a concrete example of a solution that satisfies Additivity and Recursive Invariance but not Monotonicity.

In games in characteristic function form monotonicity and additivity are not independent conditions. For games in characteristic function form, Young [50] has shown that a form of monotonicity, called strong monotonicity, can be used to characterize the Shapley Value. Indeed, this charac-

¹³ Theories that are intended for application in organizations that have longevity, where many decisions will be faced over time that will be handled the same way, should not be based on threats to defect and destroy the organization. See the remarks below in Section 1.6.3 on local and situational incentives.

terization does not make use of additivity – which was Shapley's route to this solution.

In the framework of this paper, monotonicity and additivity characterize a fairly large set of solutions – see Lemma 5. One additional axiom, Recursive Invariance, narrows the set of solutions down further to the one-parameter family discussed above. This is the main result – see Theorem 1. A natural strengthening of the continuity requirement eliminates only one of these solutions – see Theorem 2.

Theorem 3 presents the interpretation of the solutions characterized in Theorem 1 in terms of averages of the best proposals of the players, adjusting these proposals in proportion to the level of inefficiency that would be entailed. This interpretation gives rise to the tax-based formula described above.

1.6.3. Relationship to social choice theory

The type of collective decision problem considered here is very different from that in much of classical social choice theory. The problems for which our theory is appropriate are those in which the decision to be taken is isolated from other factors affecting the players involved. In contrast, social choice theory takes a more global perspective. Particular decisions are not isolated. Everything is compressed into the idea of an all-inclusive "social state". These social states can be evaluated by the players, and this evaluation is interpreted within social choice theory as the players' overall welfare level. In social choice theory, typical axioms relate to how an individual evaluates his or her own welfare. Some theories allow for comparisons of one's own welfare to that of others, or to the welfare that would be achieved if the real outcomes that are given to others were instead received by the player in question. Achieving fairness, efficiency and consistency, across both people and situations, are the goals. By abstracting from the particular details of a social decision problem, social choice theory gains its strength, generality and normative force.

In contrast to social choice theory, the theory developed here does not make reference to overall outcomes. The idea is simply to be fair "locally" – using as a reference only the outcomes relevant to the decision at hand. How well or badly situated the participants are in the remainder of their experiences is not a part of this analysis.¹⁴ The idea is only to be fair in so far as the decision at hand is concerned. For this reason, Elster [10], Sen [39], Young [51] and others have used the terms "local justice" or "situational justice" to describe the spirit of models such as the one presented in this paper.

1.6.4. Prior work on quasi-linear social choice problems

Prior work on quasi-linear social choice problems has identified several classes of solutions. With one exception, all these solutions differ from those in this paper. The exception is the solution called Equal Allocation of Non-

¹⁴ Indeed, since the "players" may not be comparable to each other comparative equity considerations may not be relevant, or even meaningful. For example, the players may be divisions of a firm with entirely distinct functional responsibilities, very different in size or in their situational incentives.

Separable Surplus (EANS), which has a long history in the context of cost allocation¹⁵. This solution corresponds to the special case of $t = 0$ in our one parameter family of solutions, as characterized in Theorem 1 and interpreted in Section 5¹⁶.

As it turns out, the EANS solution is the only member of the family we characterize that is eliminated by the strengthening of our continuity axiom in Theorem 2. All the solutions corresponding to $t \in (0, \infty)$ satisfy this stronger continuity axiom – but they will in general differ from EANS.

Both Moulin [21] and Chun [6] formulate axioms that focus on the behavior of the solutions as the number of participants vary. In their models the set of actions is fixed, although the players' evaluations of these actions can vary. Dubins [9] uses an axiom that relates to the incentive properties that solutions would possess if they were played as non-cooperative games. With the sole exception of Equal Allocation of Non-Separable Surplus, the solutions determined in these papers do not satisfy the additivity hypothesis discussed above.

In contrast to these approaches, the solutions defined in this paper are explicitly meant to apply only to two-player situations in which the outcome is determined cooperatively, with full information about the payoffs and therefore no incentive problems. The number and nature of the actions is variable across problems and is not constrained.

1.7. Outline of this paper

Section 2 presents the model. Section 3 presents the axioms and discusses them in more detail. Properties of solutions are presented in Section 4, which contains all the principal results and the proof of the main theorem. Proofs of other results follow directly from the discussion in the text. In Section 5 we interpret these results in terms of "claims", adjusted for "inefficiency" by means of "taxes", as discussed above. A geometrical construction of solutions is also given in this section. Section 6 collects various comments and comparisons to other work and discusses extensions and open problems arising from this research.

2. The model

The two players are denoted by $i = 1, 2$. A problem is a set $B \subset \mathbb{R}^2$ that is closed, convex, comprehensive and bounded above. The set of all problems is denoted $\mathcal{B}$.

For each $B \in \mathcal{B}$, let $\bar{x}(B) = \max_{x \in B} x_1 + x_2$. Let $H(B) = \{z \in \mathbb{R}^2 | z_1 + z_2 = \bar{x}(B)\}$.

A solution is a function $f : \mathcal{B} \rightarrow \mathbb{R}^2$.

The interpretation of a solution f is that when f is applied to a problem B , the final utilities received by the players are $f(B) = (f_1(B), f_2(B))$. Moreover, only the set of utilities B is relevant; the way in which these utilities are

produced from a set of underlying actions and their evaluation by the players is irrelevant.

3. Axioms

If $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ denote by $\pi\xi$ the vector $(\xi_2, \xi_1) \in \mathbb{R}^2$. Similarly, if $X \subset \mathbb{R}^2$, denote by πX the set $\{(\xi_1, \xi_2) \in \mathbb{R}^2 | \pi\xi \in X\}$.

If $X \subset \mathbb{R}^2$ and X is bounded above, define the *comprehensive hull* of X as the smallest set in $\mathcal{B}$ that contains X , and denote it by $K(X)$.

The first four axioms are entirely standard:

Axiom Efficiency (EFF): A solution f satisfies *efficiency* if for all $B \in \mathcal{B}$, $f(B) \in H(B)$.

Efficiency entails that the utility outcome $f(B)$ must be achieved by selecting an action producing $x \in B \cap H(B)$ and by making a transfer of money $t = (t_1, t_2)$, with $t_1 = -t_2$, such that $f_i(B) = x_i + t_i$, for $i = 1, 2$. If there are multiple efficient actions, any of them can be selected provided the appropriate monetary transfer is imposed so that the solution remains at $f(B)$.

Axiom Anonymity (AN): A solution f satisfies *anonymity* if for all $B \in \mathcal{B}$, $\pi f(B) = f(\pi B)$.

Axiom Continuity (CON): A solution f satisfies *continuity* if it is continuous in the Hausdorff topology on $\mathcal{B}$.

Axiom Independence of Utility Origins (IUO): A solution f satisfies *independence of utility origins* if for all $x \in \mathbb{R}^2$, and all $B \in \mathcal{B}$, $f(B + \{x\}) = f(B) + x$.

Efficiency, anonymity and continuity require no further comment. Independence of utility origins expresses the idea that the solution should depend on the relative evaluation that each individual has of the various actions. The evaluation scale used in a numerical representation of a quasi-linear utility function is determined only up to an additive constant. This constant should not affect the real aspects of the solution – the selection of the action to be taken and the transfer to be made¹⁷.

The first question one might ask is whether compensatory transfers should be paid at all. Why not simply select (the midpoint) of the set of efficient alternatives $B \cap H(B)$ and leave it at that?

This "no transfer solution" is eliminated by the four axioms above. Consider a sequence of problems with only two actions, an efficient action which remains fixed, and an inefficient action which is improved throughout the sequence until, in the limit, its payoff converges to a different point than the efficient action, but one that is equally efficient. If throughout this se-

¹⁷ Independence of Utility Origins plays the same role in our theory as Independence of Linear Transformations of Utility Scales plays in Nash-type theories. Numerical representation of the preferences, incorporating free parameters with no behavioral meaning, should not affect the real behavior results.

¹⁵ See Ransmeier [32], Straffin and Heaney [41], and Moulin [25].

quence no transfers were paid the solution would be at the efficient point. By continuity, in the limiting case it would remain at this point. However, by anonymity and independence of utility origins, the solution in the limiting case should be the average of the payoffs of the two actions. Never making any transfers would thus result in a discontinuity at this limit. Positive monetary transfers must be paid in at least some cases.

We will present three more axioms, which together with the four axioms above will characterize a one-parameter family of solutions. These axioms, Additivity, Monotonicity, and Recursive Invariance, will be imposed sequentially, in this order. At each step, we obtain a family of solutions that are easy to describe. The first axiom is well known; the second is closely related to monotonicity axioms used in other bargaining models but has not been previously applied in quasi-linear social choice theories; and the third has been mentioned in the literature only once before, but has not been used in a characterization theorem¹⁸.

Axiom Additivity (ADD): A solution f satisfies *additivity* if for all $B_1, B_2 \in \mathcal{B}$, $f(B_1 + B_2) = f(B_1) + f(B_2)$.

Additivity is commonly used in games in characteristic function form, where it characterizes the Shapley Value. A related axiom, called *superadditivity*, is used in Nash Bargaining Theory, where for the two-person case it characterizes the Maschler-Perles [19] solution.¹⁹

The justification for additivity is based on considering those special pairs of problems that do not interact at all. In such pairs, preference over actions in each problem are invariant to the action taken in the other. Pairs of non-interacting problems can be combined into a composite problem. Each action in the composite problem is actually a pair of actions, one chosen in each of the original problems. Additivity is simply the requirement that the players should be indifferent as to whether independent problems are solved separately or jointly. Further justification for this axiom is presented in Myerson [28], Roth [34], Maschler and Perles [19] and Peters [30].

Axiom Monotonicity (MON): A solution f satisfies *Monotonicity* if $f_1(K(\{(0,0)\} \cup \{(1,-x)\}))$ is non-increasing in x , for $x \geq 1$.

Monotonicity is related in spirit to various monotonicity axioms in the bargaining literature²⁰, but has different implications arising from the structure of quasi-linear social choice problems. The intent of this axiom is to capture the ethical spirit of a process in which monetary transfers are paid to a player who can make a good argument, to a neutral observer, that he has "forgone" the prospect of decisions that would have been better for him, in favor of one of the

¹⁸ See footnote 7 above.

¹⁹ However, this solution exists only for the case of $n = 2$, see Maschler and Perles [20]. On the other hand, in the present model, the axioms EFF, AN, CON, IUO, and ADD can be satisfied for any n . Indeed there are infinitely many solutions that do so, see Green [11].

²⁰ See Kalai-Smorodinsky [16], Moulin [25], Roth [36], Thomson and Myerson

collectively efficient decisions. Such an argument should be stronger when the forgone alternative is less inefficient, but no less beneficial to this player.

Now we come to the final axiom:

Axiom Recursive Invariance (RI): A solution satisfies *recursive invariance* if for all $B \in \mathcal{B}$, $f(K(B \cup \{f(B)\})) = f(B)$.

Recursive Invariance is based on the idea that when players agree to use a solution f they are committing themselves to a process that imposes monetary transfers in order to compensate each other for forgoing alternatives that would have been preferred but which might not be efficient. The players are agreeing to make a transfer that results in $f(B)$ recognizing that this allocation may not be feasible through the choice of an action alone. Recursive Invariance embodies the idea that the players already regard $f(B)$ as a fair outcome when the underlying possibilities are B . Having committed themselves to a process that results in this outcome, the players should still consider $f(B)$ to be a just result if a means of achieving it directly, without the use of transfers, were added to the feasible outcomes already in B .

Recursive Invariance can also be interpreted as a form of renegotiation proofness condition. Imagine that after B becomes known, but prior to the actual implementation of any outcome or transfers, the players enter into a renegotiation. At this stage, given that the players have agreed to use the solution f , $f(B)$ has the same status as all the points in the original B – it is an outcome that can be selected without the need for any *further* monetary transfers. If $f(K(B \cup \{f(B)\})) \neq f(B)$ the players would be rejecting the recommendation of the solution f at this stage, renegotiating their original commitment to use f even though nothing real has changed.

4. Constructing solutions

The construction of solutions is based on a simple geometrical idea. Problems in $\mathcal{B}$ can all be written as the sum of sets in a very simple subfamily – the subfamily of $\mathcal{B}$ in which there are only two outcomes. Once the solution is fixed on this subfamily, it can be extended to all of $\mathcal{B}$ using Additivity.

Monotonicity and Recursive Invariance restrict the behavior of solutions on this subfamily. These restrictions have implications for the behavior of solutions on all of $\mathcal{B}$ which are expressed in the main characterization theorems.

To implement this approach to the construction of solutions, some additional notation will be useful.

Let $\mathcal{B}_0 \subset \mathcal{B}$ be the set of $B \in \mathcal{B}$ with $\bar{x}(B) = 0$, and $(0,0) \in B$. For $x \geq \lambda > 0$, let $C_1(\lambda, x) = K(\{(0,0)\} \cup \{(\lambda, -x)\})$ and $C_2(\lambda, x) = K(\{(0,0)\} \cup \{(-x, \lambda)\})$. The solutions on the subfamily consisting of all sets $C_i(\lambda, x)$ for $i = 1, 2$ – problems with only two outcomes – will determine the solutions on $\mathcal{B}_0$. These solutions can then be extended to all of $\mathcal{B}$. The basic idea is simple: One can approximate an arbitrary problem $B \in \mathcal{B}_0$ by a polyhedral problem. Recognize that such a polyhedral problem will be the

these alternatives is always $(0, 0)$. Thus, once we decide how to solve these two alternative problems, Additivity tells us how the solution must behave on polyhedral problems and Continuity tells us how to extend it to $\mathcal{B}_0$. To go from $\mathcal{B}_0$ to all of $\mathcal{B}$, we use the translation invariance of the solutions guaranteed by Independence of Utility Origins, and invoke Additivity once again.

Stopping at this point gives us all the additive solutions. As we will see, it is a very large family. Anonymity, Monotonicity, and Recursive Invariance enable us to reduce it to the one parameter family described above.

The first three lemmas give precise statements of construction of the additive solutions.

Any $B \in \mathcal{B}_0$ can be approximated by a finite sum of the form $\sum_{j=1, \dots, J} C_1(\lambda_j^1, x_j^1) + \sum_{j=1, \dots, J} C_2(\lambda_j^2, x_j^2)$. An exact decomposition of $B \in \mathcal{B}_0$ as a sum of sets each of which is generated by a problem with only two actions is given by:

Lemma 1 For each $B \in \mathcal{B}_0$ there exists a pair of non-negative measures μ_1^B, μ_2^B on $[1, \infty)$ such that

$$B = \int C_1(1, x) d\mu_1^B(x) + \int C_2(1, x) d\mu_2^B(x)$$

Anonymity and additivity give rise to the translation invariance of solutions, which can be expressed as:

Lemma 2 If f satisfies *EFF*, *AN*, and *IUO*, $f(K(\{x\})) = x$ for all $x \in \mathbb{R}^2$

As a consequence of Lemmas 1 and 2, given the value of a solution on the sets $C_i(1, x)$, for $i = 1, 2$ and $x \geq 1$, we can find the solution to any problem $B \in \mathcal{B}$ by translating B by a vector $-x$ such that $x \in B \cap H(B)$, resulting in a set $B_0 \in \mathcal{B}_0$. Then, the solution $f(B_0)$ is computed using Lemma 1, and the original problem B is therefore solved at $f(B_0) + x$. This process is summarized in Lemma 3:

Lemma 3 If $f : \mathcal{B}_0 \rightarrow \mathbb{R}^2$ satisfies *EFF*, *AN*, *IUO* and *ADD* on $\mathcal{B}_0$, then f can be uniquely extended to $\mathcal{B}$.

Given a solution f let $g^f : [1, \infty) \rightarrow \mathbb{R}$ be defined by

$$g^f(x) = f_1(C_1(1, x))$$

Given a function $g : [1, \infty) \rightarrow \mathbb{R}$, Lemma 1 implies that g generates a solution f via the relation:

$$f_1(B) = \int g(x) d\mu_1^B(x) - \int g(x) d\mu_2^B(x) \quad \text{for all } B \in \mathcal{B}_0$$

We can characterize a family of solutions $\mathcal{F}$ that are consistent with *AN*, *CON*, *ADD* and any additional axioms, by determining the properties that the associated g^f must satisfy for any $f \in \mathcal{F}$ under these axioms.

Lemma 4 Given any continuous function $g : [1, \infty) \rightarrow \mathbb{R}$, such that $g(1) = \frac{1}{2}$, there exists a unique solution f satisfying *EFF*, *AN*, *CON*, *IUO* and *ADD* such that $g^f(x) = g(x)$ for all $x \in [1, \infty)$. Conversely, if f is a solution satisfying *EFF*, *AN*, *CON*, *IUO* and *ADD* then g^f is continuous and $g^f(1) = \frac{1}{2}$.

Lemma 5 If f is a solution satisfying *EFF*, *AN*, *CON*, *IUO*, *ADD* and *MON*, then g^f is continuous, non-increasing and $g^f(1) = \frac{1}{2}$. Conversely, if $g : [1, \infty) \rightarrow \mathbb{R}$ is any continuous, non-increasing function with $g(1) = \frac{1}{2}$, the solution $f : \mathcal{B} \rightarrow \mathbb{R}^2$ that is generated from g , will satisfy *EFF*, *AN*, *ADD*, *CON*, *IUO* and *MON*.

Lemma 5 follows directly from the statement of the monotonicity axiom.

The main result of this paper is the characterization of the set of solutions satisfying Recursive Invariance, in addition to the axioms previously imposed:

Theorem 1 If f is a solution satisfying *EFF*, *AN*, *CON*, *IUO*, *ADD*, *MON* and *RI*, $g^f : [1, \infty) \rightarrow \mathbb{R}$ satisfies either:

(i) There exists $x^* > 1$ such that

$$g^f(x) = \begin{cases} \frac{1}{2} - \frac{1}{2} \left(\frac{x-1}{x^*-1} \right) & \text{for } x \leq x^* \\ 0 & \text{for } x > x^* \end{cases}$$

or

(ii) $g^f(x) = \frac{1}{2}$ for all $x \in [1, \infty)$.

Conversely, all the solutions generated from functions g satisfying (i) or (ii) are consistent with the axioms *EFF*, *AN*, *CON*, *IUO*, *MON* and *RI*.

Proof of Theorem 1

If $g(y) \equiv \frac{1}{2}$ we have (ii) in the statement of the theorem. If there exists y such that $g(y) > \frac{1}{2}$, we have a violation of Monotonicity. Assume therefore that there exists $y > 1$ such that $0 < g(y) < \frac{1}{2}$. By the definition of g we have,

$$f(K(\{(0, 0)\} \cup \{(1, -y)\})) = (g(y), -g(y)) \quad (1)$$

By *RI*,

$$f(K(\{(0, 0)\} \cup \{(g(y), -g(y))\} \cup \{(1, -y)\})) = (g(y), -g(y)) \quad (2)$$

Now we use *ADD* to express the argument of f on the left hand side of (2) as the sum of two sets in $\mathcal{B}_0$:

$$\begin{aligned} K(\{(0, 0)\} \cup \{(g(y), -g(y))\} \cup \{(1, -y)\}) \\ = K(\{(0, 0)\} \cup \{(g(y), -g(y))\}) + K(\{(0, 0)\} \cup \{(1 - g(y), -y + g(y))\}) \end{aligned} \quad (3)$$

By *IUO* and *AN*

$$f(K(\{(0, 0)\} \cup \{(g(y), -g(y))\})) = \left(\frac{g(y)}{2}, -\frac{g(y)}{2} \right) \quad (4)$$

Substituting (3) into (2) and using (4) we have

$$f(K(\{(0,0)\} \cup \{(1-g(y), -y+g(y))\})) = \left(\frac{g(y)}{2}, -\frac{g(y)}{2}\right) \quad (5)$$

Now apply the same argument to

$$K(\{(0,0)\} \cup \left\{\left(\frac{g(y)}{2}, -\frac{g(y)}{2}\right)\right\} \cup \{(1-g(y), -y+g(y))\}) \quad (6)$$

Decompose the argument of (6) as in (3):

$$\begin{aligned} & K(\{(0,0)\} \cup \left\{\left(\frac{g(y)}{2}, -\frac{g(y)}{2}\right)\right\} \cup \{(1-g(y), -y+g(y))\}) \\ &= K(\{(0,0)\} \cup \left\{\left(\frac{g(y)}{2}, -\frac{g(y)}{2}\right)\right\}) \\ &\quad + K\left(\{(0,0)\} \cup \left\{\left(1-g(y) - \frac{g(y)}{2}, -y+g(y) + \frac{g(y)}{2}\right)\right\}\right) \end{aligned} \quad (7)$$

Therefore,

$$f_1\left(K\left(\{(0,0)\} \cup \left\{\left(1-g(y) - \frac{g(y)}{2}, -y+g(y) + \frac{g(y)}{2}\right)\right\}\right)\right) = \frac{g(y)}{4} \quad (8)$$

Recursively,

$$f_1\left(K\left(\{(0,0)\} \cup \left\{\left(1-g(y)\left(2 - \frac{1}{2^n}\right), -y+g(y)\left(2 - \frac{1}{2^n}\right)\right\}\right)\right)\right) = \frac{g(y)}{2^{n+1}} \quad (9)$$

Taking the limit as $n \rightarrow \infty$, we have by CON,

$$f_1(K(\{(0,0)\} \cup \{(1-2g(y), -y+2g(y))\})) = 0 \quad (10)$$

or

$$(1-2g(y))g\left(\frac{y-2g(y)}{1-2g(y)}\right) = 0. \quad (11)$$

Now, since $0 < g(y) < \frac{1}{2}$,

$$g\left(\frac{y-2g(y)}{1-2g(y)}\right) = 0. \quad (12)$$

Let

$$x^* = \frac{y-2g(y)}{1-2g(y)}. \quad (13)$$

Taking n large and recalling that $1 < y$ and $0 < g(y) < \frac{1}{2}$ (9) implies that $g(x) > 0$ for a sequence of points x^n approaching x^* from below. Therefore, by MON,

$$x^* = \inf\{x | g(x) = 0\} \quad (14)$$

Beginning this argument from any y such that $g(y) > 0$, we see that (13) holds independent of the value of y selected. Thus,

$$x^* = \frac{x-2g(x)}{1-2g(x)} \text{ for all } x < x^* \quad (15)$$

Solving (15) over its domain of validity, we obtain,

$$g(x) = \frac{1}{2} - \frac{x-1}{2(x^*-1)} \text{ for all } x < x^* \quad (16)$$

Thus, for the case where there exists y with $g(y) < \frac{1}{2}$, it remains only to show that $g(x) = 0$ for $x > x^*$. Suppose to the contrary that there exist $y > x^*$ with $g(y) < 0$. Now (2) holds independent of the sign of $g(y)$. However, because $g(y) < 0$,

$$\begin{aligned} & K(\{(0,0)\} \cup \{(g(y), -g(y))\} \cup \{(1, -y)\}) \\ &= K(\{(0,0)\} \cup \{(g(y), -g(y))\}) + K(\{(0,0)\} \cup \{(1, -y)\}). \end{aligned} \quad (7)$$

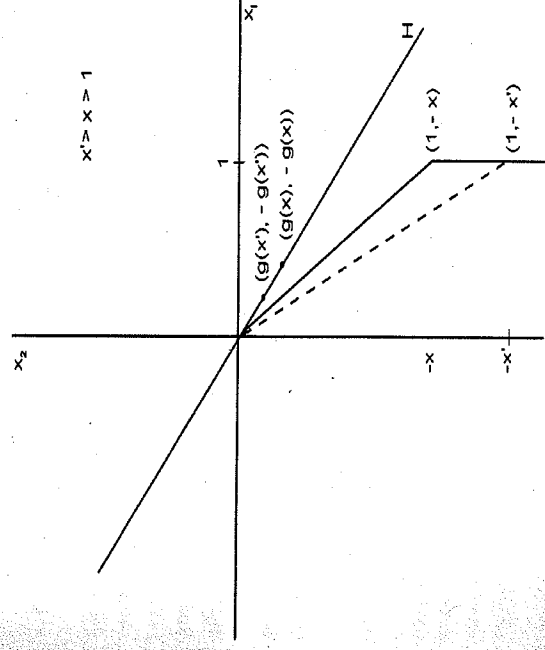
Applying ADD and AN,

$$\begin{aligned} & f(K(\{(0,0)\} \cup \{(g(y), -g(y))\} \cup \{(1, -y)\})) \\ &= \left(\frac{g(y)}{2}, -\frac{g(y)}{2}\right) + (g(y), -g(y)) \end{aligned} \quad (18)$$

which contradicts (2). ■

Continuity as stated is based on the idea that if payoffs from a given set of actions are perturbed slightly, there should be only a slight change in the resulting utilities. We will now introduce a slight strengthening of the continuity idea with a different motivation: Actions that result in extremely inefficient payoffs should have little weight in determining the outcome. The mathematical way of expressing this idea is as follows:

Take a sequence of sets B^k . If there exists $B \in \mathcal{B}$ such that for every compact set $X \subset \mathbb{R}^2$, $B^k \cap X$ converges in the Hausdorff topology to $B \cap X$, we will say that B^k converges to B . A topology, which we call the bounded convergence topology, is defined by this family of converging sequences.



Continuity can be strengthened to:

Axiom Continuity* (CON*): The solution f is continuous in the bounded convergence topology.

Lemma 6 If f is a solution satisfying EFF, AN, CON*, and IUO, then the function g^f that represents f satisfies: $\lim_{x \rightarrow \infty} g^f(x) = 0$

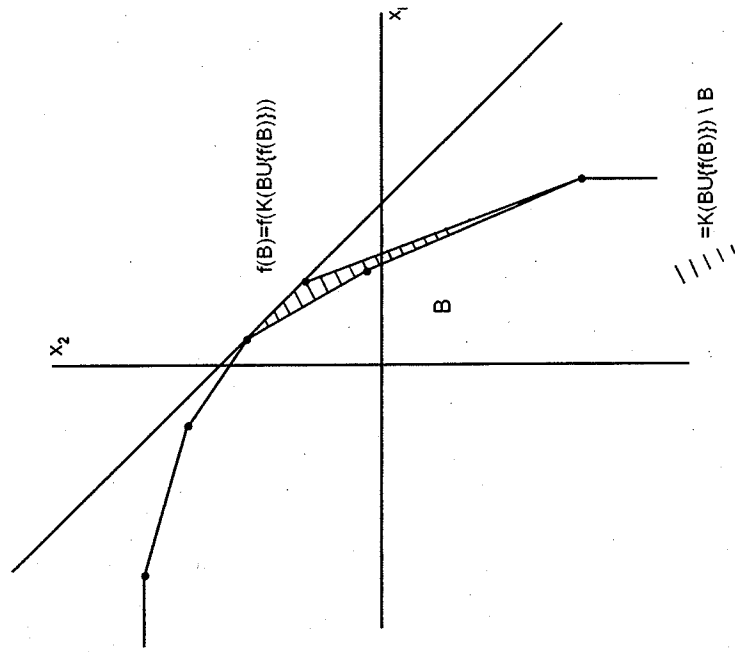
Theorem 2 If f is a solution satisfying EFF, AN, CON*, IUO, ADD, MON and RI, then there exists $x^* \in (1, \infty)$ such that $g^f: [1, \infty) \rightarrow \mathbb{R}$ satisfies:

$$g^f(x) = \begin{cases} \frac{1}{2} - \frac{1}{2} \left(\frac{x-1}{x^*} \right) & \text{for } x \leq x^* \\ 0 & \text{for } x > x^* \end{cases}$$

5. Characterizations of solutions in terms of taxes on the inefficiency of claims

In this section we offer an interpretation of Theorem 1 in terms of "best claims for compensation", as discussed in the introduction.

For each $x^* > 1$ and $x \geq 1$, let $g^x(x) = \max(\frac{1}{2} - \frac{1}{2}(\frac{x-1}{x^*}), 0)$. Define the solution $f(B; x^*)$ as the solution generated from g^x by $f_1(B, x^*) = \int g^x(x) d\mu_1^B(x) - \int g^x(x) d\mu_2^B(x)$.



Take any $x^0 \in B \cap H(B)$. For each $t \geq 0$ define

$$f^{t, x^0}(B) = x^0 + (1/2)[(y_1^*(B, t), -y_1^*(B, t)) + (-y_2^*(B, t), y_2^*(B, t))]$$

where $y_i^*(B, t)$ are the value of the problems

$$\max_{x \in B} (x_i - x_i^0) - t(\bar{x}(B) - (x_1 + x_2))$$

for $i = 1, 2$.

Lemma 7 The solution $f^{t, x^0}(B)$ is independent of the choice of $x^0 \in B \cap H(B)$.

Therefore we can define $f^t(B) = f^{t, x^0}(B)$.

Theorem 3 For each $x^* > 0$, the solution $f(B, x^*)$ is equal to the solution $f^t(B)$ for $t = \frac{1}{x^* - 1}$.

The solution generated from $g(x) \equiv \frac{1}{2}$ is equal to the solution $f^0(B)$.

The limiting case, where t is 0, corresponds to $x^* \rightarrow \infty$ and part (ii) of the conclusion of Theorem 1: $g^f(x) \equiv 1/2$. In this solution, no decrease in the claim of either player is applied due to the inefficiency of a superior forgone alternative. This is the Equal Allocation of Non-Separable Surplus solution. (See the further discussion in Section 6.1 below.)

The geometry of the solutions are shown in Figure 3. The efficient point is $x^0 = (0, 0)$. Both of the players have superior forgone alternatives. If their payoffs in these two alternatives are adjusted in accordance with the "tax" t , their "best claims" for compensation are the points y^1 and y^2 . The solution, therefore is $f(B) = x^0 + \frac{1}{2}(y^1 + y^2)$, as indicated.

6. Other remarks

6.1. Comparison to other solutions

Moulin [21] discusses three classes of solutions to quasi-linear social choice problems. The first is simply to choose an efficient action and to make no transfers at all. This method necessarily produces discontinuities in the utility outcomes in the neighborhood of problems where there is more than one efficient decision. (See the discussion above in Section 3.)

The second class of solutions is called Equal Sharing Above a Convex Reference Level²¹. These methods produce outcomes that depend on the description of the problem in ways other than the set of utilities reached by an action choice. For example, adding other actions whose induced utilities duplicate those already in the feasible set will affect the solution.

The third solution is called, in the cost allocation literature, Equal Allocation of Non-Separable Costs. The EANS solution is obtained by computing best results that can be attained by coalitions of $n - 1$ players if they could choose the action without reference to the results of the omitted player: Define $v_{-i} = \max_{x_{-i} \in B} \sum_{j \neq i} x_j$. Then the separable cost (or benefit) ascribed to

The one parameter family of solutions studied above satisfy the natural generalization of all the axioms in this non-zero cost framework. Any efficient action can be selected, resulting in the benefits x_i^* and the cost $-x_0^*$. Begin from the payoffs where the cost at the efficient action are shared equally, resulting in payoffs $x_i^* + \frac{x_0^*}{2}$. Then, compute the claim y_i^* of each player according to the value of the problem

$$\max_{x \in B} \left(x_i + \frac{x_0}{2} \right) - \left(x_i^* + \frac{x_0^*}{2} \right) - t(\bar{x}(B) - (x_0 + x_1 + x_2))$$

The solution, at the parameter value t is $(f_1^t(B), f_2^t(B)) = (x_1^* + \frac{x_0^*}{2} + \frac{y_1 - y_2}{2}, x_2^* + \frac{x_0^*}{2} + \frac{y_2 - y_1}{2})$.

6.4. More than two players

Unlike the extension of our basic model to problems with non-zero costs, the extension to more than two players is anything but trivial. The problem is not that there are no solutions, but rather that there may be others that satisfy the axioms as well.

Consider the subfamily of problems in which all actions are equally efficient and where the maximum possible for the group as a whole is zero. In the case of $n = 2$, these problems are trivial: The solution is the midpoint of the efficient segment of B .

If we were to apply the idea of the solutions obtained above to the case of $n > 2$, the value of t would be irrelevant because all alternatives are equally efficient, and all solutions would give player i the payoff $\frac{1}{n} \sum_{j \neq i} \max_B x_j$.

The solutions that reach this payoff are not, however, the only solutions that satisfy the additivity axiom on this family of problems. One example of a solution that can take a different value on some problems is obtained by defining a game v in characteristic function form as $v(S) = \max_{x \in B} \sum_{i \in S} x_i$ and letting the solution be the Shapley value of v . Indeed, the results in Green [11] show that there are an infinity of values that can be taken on by additive solutions in this class of problems.

The reason for this difference in the results can be traced to the method of proof used above in the case of $n = 2$. We decomposed problems in $\mathcal{B}$ into a sum of problems involving only two actions (recall Lemma 1). Therefore to find solutions satisfying Additivity we needed only to specify the solutions on the very simple subfamily of "two-action" problems, which is the role of the function g above. Once we generated all additive solutions from this function of one variable, the other axioms were used to restrict g . It is this idea that does not generalize to larger n . In spaces of larger dimension a typical member of $\mathcal{B}$ cannot be written as the sum of such a very simple family of problems. There is, therefore, quite a bit more flexibility available to the "design" of an additive solution.²³

²² Using the constant sum property of the feasible set, i 's payoff can also be written as $\min_B x_i - \frac{1}{n} \sum_{j=1, \dots, n} \min_B x_j$.

²³ Mathematically, the question is related to problems in the theory of convex polytopes. See

Monotonicity and Recursive Invariance deal with problems in which not all actions are efficient. At present it is not known whether all members of the class of solutions identified by Green [11] for the "all-actions-efficient" case can be extended to the general case in a way that preserves Monotonicity and Recursive Invariance. This remains one of the principal questions for further research in this area.

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