BILATERAL CONTRACTS*

Jerry GREEN

Harvard University, Cambridge, MA 02138, USA
N.B.E.R., Cambridge, MA 02138, USA

Seppo HONKAPOHJA

University of Helsinki, 00170 Helsinki, Finland
Yrjo Jahnsson Foundation, Helsinki, Finland

A mathematical characterization of self-enforcing bilateral contracts is given. Contracts where both parties exercise some control over the quantity traded can sometimes be superior to contracts that rest control entirely with one side. Some qualitative characteristics of these contracts are given.

1. Introduction

The basic form of economic exchange is a bilateral relationship between buyer and seller. If economic conditions are common knowledge, there is no problem in principle to find the efficient quantity to trade. But if benefits are known only to the buyer and costs are known only to the seller a bargaining situation results. In such circumstances economic efficiency might be improved if a contract governing the transaction could be agreed upon in advance. Such a contract would give control of various aspects of the exchange to the two parties involved. This paper studies contracts of this nature. We examine the feasibility of implementing various agreements.

One approach to this problem is to give control completely to one party or the other. This is seen widely in practice as well as in theory.[1] A price per unit may be fixed and the buyer can name his quantity after seeing the actual

---

    [1]Weitzman (1974).

benefits that are relevant. A more complex version presents the buyer with a non-linear price schedule.[2] Alternatively, the supplier may be given control in a contract with a specified revenue function along which he can optimize.

These solutions are fairly well understood. When the uncertainty is entirely or primarily on one side of the market they can duplicate the fully efficient solution — that is the quantity that would be traded in a full-information world. When the random influences impact both parties significantly, full efficiency is not attainable. The choice of which side should govern the contract is then dependent on the elasticity of benefits and costs, and on the distribution of the random parameters.

The primary goal of this paper is to examine contracts that allow for mutual control. While these contracts do not have the ability to achieve the first-best, they may, in some cases, dominate one-sided governance.

In the next section the basic model is set out. It is shown that feasible contracts lead to traded quantities which, viewed as functions of the random parameters, have to satisfy a certain partial differential equation.

Section 3 examines the special case in which the contract is so arranged as to be equally beneficial to the two parties in all circumstances. This case allows us to restrict the partial differential equation in section 2, obtaining a second-order ordinary differential equation. Because this equation is non-linear and because its right-hand side diverges at some points, its solutions divide naturally into several different types. These are studied in sections 4 and 5, and qualitative properties implied by them are presented in section 6.

Numerical methods are used in section 7 to calculate various solutions whose properties are shown to accord with the theory.

2. The model

The basic structure of the model follows Weitzman (1974). There is a buyer, whose willingness to pay for the good is

$$U(q) = \tfrac{1}{2}aq^2 + \varepsilon q, \tag{1}$$

where q is the quantity traded. The other party to the contract is the seller, whose reservation value for q is the negative of

$$V(q) = \tfrac{1}{2}bq^2 - \delta q. \tag{2}$$

Concavity requires that a and b be negative.

All of the uncertainty in the model enters through the coefficients of the linear terms, ε and δ. It is useful to note at the outset that the efficient

[2]Spence (1977).

quantity is that which maximizes $U + V$,

$$q^*(\varepsilon, \delta) = -(\varepsilon - \delta)/(a + b). \tag{3}$$

To insure that q^* is positive we suppose that $\varepsilon > \delta$ with probability one. In order to use the methods of incentive compatibility, it is convenient to assume that (ε, δ) has a continuous bivariate distribution over a rectangle in R^2.

A *contract* is a pair of functions t, q which assign to each (ε, δ) the monetary payment made by the buyer to the seller, $t(\varepsilon, \delta)$, and the quantity received by the buyer from the seller $q(\varepsilon, \delta)$. Given any contract and given the realized values of ε and δ the two players can be viewed as participants in a game where the strategies are their professed values of ε and δ, $\tilde{\varepsilon}, \tilde{\delta}$, and their payoff functions are, respectively,

$$U(q(\tilde{\varepsilon}, \tilde{\delta})) - t(\tilde{\varepsilon}, \tilde{\delta}), \tag{4}$$

and

$$V(q(\tilde{\varepsilon}, \tilde{\delta})) + t(\tilde{\varepsilon}, \tilde{\delta}). \tag{5}$$

Viewed in this way, contracts are *direct revelation mechanisms* in the sense of Green–Laffont (1979) or Laffont–Maskin (1980). We will say that a contract is *self-enforcing* or *incentive compatible* if the true value of ε and the true value of δ, respectively, dominant strategies in this game for the buyer and seller, respectively. If a contract were not self-enforcing, the value to the two players would have to be computed at the equilibria of the game. Multiple equilibria would typically arise. Little is known about this case. In this paper we examine self-enforcing contracts exclusively.[3] Moreover, as a technical matter it is convenient to assume that $q(\cdot, \cdot)$ and $t(\cdot, \cdot)$ are twice continuously differentiable.[4]

We now give a characterization of self-enforcing contracts. The optimal strategies for players whose true parameters are $\bar{\varepsilon}$ and $\bar{\delta}$ are determined by the first-order conditions

$$aq(\varepsilon, \delta)q_\varepsilon(\varepsilon, \delta) + \bar{\varepsilon}q_\varepsilon(\varepsilon, \delta) - t_\varepsilon(\varepsilon, \delta) = 0, \tag{6}$$

and

$$bq(\varepsilon, \delta)q_\delta(\varepsilon, \delta) - \bar{\delta}q_\delta(\varepsilon, \delta) + t_\delta(\varepsilon, \delta) = 0, \tag{7}$$

---

[3]It may indeed be the case that by using strategy spaces other than the real line, we can implement *discontinuous* $q(\cdot, \cdot)$ which nevertheless have a lower welfare loss. This is beyond the scope of the present paper.

[4]To be precise, differentiability at every point in the domain will be required. We do not consider discontinuous, piecewise differentiable or other weaker solution concepts. However, this assumption is slightly relaxed in section 5.

where subscripts denote partial differentiation. Incentive compatibility requires that these be identities in (ε, δ) when evaluated at $\varepsilon = \bar{\varepsilon}$ and $\delta = \bar{\delta}$.

Differentiating (6) with respect to δ and (7) with respect to ε, we find (suppressing the arguments of all functions)

$$aqq_{\varepsilon\delta} + aq_\delta q_\varepsilon + \varepsilon q_{\varepsilon\delta} - t_{\varepsilon\delta} = 0, \tag{8}$$

and

$$bqq_{\varepsilon\delta} + bq_\varepsilon q_\delta - \delta q_{\varepsilon\delta} + t_{\delta\varepsilon} = 0. \tag{9}$$

Using $t_{\varepsilon\delta} = t_{\delta\varepsilon}$ we can eliminate $t_{\varepsilon\delta}$ from (8) obtaining

$$((a+b)q + (\varepsilon - \delta))q_{\varepsilon\delta} + (a+b)q_\varepsilon q_\delta = 0 \tag{10}$$

Eq. (10) is the fundamental partial differential equation of this theory of bilateral contracts.

Note that *any* function of only one of the two variables will satisfy (10). This is another way of seeing that one-sided contract governance can be made quite flexible by choosing the non-linear price or revenue functions appropriately.[5] The first-best given in (3), however, is unattainable through any self-enforcing scheme.[6]

Before specializing and examining the nature of the solutions to (10), which will be the subject of the rest of this paper, two further points should be made. The individuals' second-order conditions must hold at each value of the parameters, and this entails some further constraints on the functions $q(\cdot, \cdot)$ that can be implemented. For the buyer, we have that

$$a(q_\varepsilon)^2 + aqq_{\varepsilon\varepsilon} + \bar{\varepsilon}q_{\varepsilon\varepsilon} - t_{\varepsilon\varepsilon} \leqq 0. \tag{11}$$

To express this as a constraint on q, note that as (6) is an identity with $\varepsilon = \bar{\varepsilon}$ we can differentiate it with respect to ε. Taking the result and subtracting it from (11) yields

$$q_\varepsilon \geqq 0. \tag{12}$$

Similarly, the seller's second-order condition when combined with the first-order conditions for all δ yields

$$q_\delta \leqq 0. \tag{13}$$

---

[5]Second-order conditions for the individuals must be respected as constraints. See below.
[6]This result is well known; see Green–Laffont (1979).

3. Solutions that divide the surplus evenly

In this paper we examine the family of self-enforcing contracts that have the additional property of dividing the sum of the surpluses equally between the buyer and the seller for all realizations of the random parameters. We know that all self-enforcing contracts satisfy (10). In addition, this equal division property can be expressed as the equality of (4) and (5). Differentiating this identity with respect to ε and using (6) gives us

$$q(\varepsilon, \delta) = [(a + b)q(\varepsilon, \delta) + (\varepsilon - \delta)]q_\varepsilon. \tag{14}$$

Differentiating it with respect to δ, using (7) and equating the result to (14) we obtain

$$(q_\varepsilon + q_\delta)[(a + b)q + (\varepsilon - \delta)] = 0. \tag{15}$$

From (15) we see that either q is the first-best, in which case the bracketed expression is zero, or else $q_\varepsilon + q_\delta = 0$. But we already know that the first-best cannot be achieved in general, that is identically in any neighborhood of (ε, δ). Thus, $q_\varepsilon + q_\delta = 0$ almost-everywhere. Differentiating with respect to ε and with respect to δ and eliminating $q_{\varepsilon\delta}$ we obtain the result that

$$q_{\varepsilon\varepsilon} - q_{\delta\delta} = 0,$$

which is the wave-equation in R^1 [see, e.g., Hellwig (1960, p. 11)]. Its solutions are

$$q(\varepsilon, \delta) = w_1(\varepsilon + \delta) + w_2(\varepsilon - \delta),$$

where w_1 and w_2 are arbitrary functions in C^2. However, from $q_\varepsilon + q_\delta = 0$ we see that w_1 must be a constant function. Therefore the equal division constraint is $q(\varepsilon, \delta) = \phi(\varepsilon - \delta)$, for some ϕ.

Let us define

$$x \equiv \varepsilon - \delta, \qquad \Psi(x) \equiv \Psi(\varepsilon - \delta) \equiv (a + b)\phi(\varepsilon - \delta) + (\varepsilon - \delta). \tag{16}$$

The basic partial differential equation (10) takes the form of the ordinary differential equation,

$$\Psi''\Psi + (1 - \Psi')^2 = 0. \tag{17}$$

The function Ψ has a straightforward economic interpretation,

$$\Psi = (a + b)(q - q^*), \tag{18}$$

i.e., it is proportional to the deviation of the quantity resulting from the contract from the first-best quantity. The second-order conditions (12) and (13) impose the constraint

$$\Psi' \leqq 1, \tag{19}$$

which, as we will see below, allows us to restrict the class of solutions to (17) corresponding to implementable contracts. The non-negativity $q(\cdot, \cdot) \geqq 0$ implies

$$\Psi \leqq x, \tag{20}$$

according to the conventions (16).

Eq. (17) is an interesting sort of differential equation for several reasons. It has one obvious family of solutions, namely,

$$\Psi(x) = x + c, \tag{21}$$

for any real number $c \leqq 0$. In economic terms these are the trivial solutions for, using (16), one can see that they correspond to $q(\varepsilon, \delta) = c/(a+b)$ — a completely inflexible and uncontingent contract.

There are other solutions to (17), and it is on these that we shall focus. The difficulty in finding some of these solutions can be traced to the fact that it does not define a unique value of Ψ'' when $\Psi = 0$. As is well-known in the theory of differential equations, the existence and uniqueness of a solution of an equation of order n, given n initial conditions, is guaranteed in a neighborhood of the initial point only if the equation is Lipschitzian throughout such a neighborhood. The irregularity in this equation occurs at a particularly unfortunate value, $\Psi = 0$, which is precisely where $q = q^*$.

Because of this fact, we will have to discuss solutions other than those given by (21) in two separate cases: those where Ψ has one-sign throughout the range of x, and those where Ψ is zero for some x. These will be called *one-signed* and *two-signed* contracts, respectively, and are analyzed separately in sections 4 and 5.

We are looking for solutions to (17) over the domain of x that could possibly arise. It is not necessary that the solution be extendable over the whole real line.[7] We will see that the solutions other than (21) indeed have the character that they cannot be extended beyond a bounded interval.

Let us consider, at first informally, the qualitative nature of solutions where Ψ has one sign. Suppose we are looking for a solution on $[x_0, x_1]$ and that we set $\Psi'(x_0) \leqq 1$, as required by the second-order conditions, and

---

[7]Indeed we have already assumed that (ε, δ) lies in a rectangle in R^2.

$\Psi(x_0) > 0$. From (17) we can see that $\Psi''(x_0)$ must be negative. Therefore Ψ' decreases further with x. At some x, $\Psi' = 0$ and Ψ is at a maximum; beyond this Ψ begins to decrease. In this region Ψ'' is diverging towards $-\infty$ because $(\Psi' - 1)^2$ is growing and Ψ is going towards zero. Such a solution exists only on intervals where the upper endpoint is below the point where this degeneracy occurs.
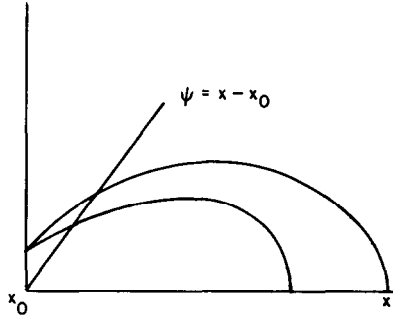

Fig. 1. Solutions with $\Psi'(x_0) < 1$, $\Psi(x_0) > 0$, for two different values of $\Psi'(x_0)$.

The requirement that the solution exist throughout the range of x can be viewed as placing constraints on $\Psi'(x_0)$ and $\Psi(x_0)$. If, for example, the degeneracy were to occur before x_1, then the value specified for $\Psi(x_0)$ could be increased. It is easy to see that the resulting trajectory would be everywhere higher and would have a degeneracy at a larger value of x_1. In this way the domain of the solution can be extended. It can also be extended by raising $\Psi'(x_0)$, which has a qualitatively similar effect.

4. One signed-solutions to (17)

In order to obtain manageable expressions for the solution, which will eventually be obtained only by numerical methods, it is useful to convert (17) into a first-order equation. This can be done because (17) is autonomous — x does not appear explicitly. Let

$$z(\Psi) = \Psi', \tag{22}$$

so that

$$\frac{d\Psi'}{dx} = \Psi'' = \frac{dz}{d\Psi}\frac{d\Psi}{dx} = \frac{dz}{d\Psi} = \frac{dz}{d\Psi} \cdot z. \tag{23}$$

Thus (17) can be rewritten as the following first-order equation for z:

$$z\frac{dz}{d\Psi}\Psi+(1-z)^2=0. \tag{24}$$

This equation is separable in z and Ψ and can be integrated to yield

$$\log\Psi=\frac{-1}{1-z}-\log(1-z)+K, \tag{25}$$

where K is a constant of integration.

This can be rewritten using (22)

$$\Psi=\frac{K}{1-\Psi'}e^{-1/(1-\Psi')}. \tag{26}$$

Thus (17) has been rewritten as a first-order equation, still autonomous, but rather non-linear in Ψ'. Such equations may be transformed[8] via differentiation with respect to x,

$$\Psi'=\frac{-K\Psi'}{(1-\Psi')^3}e^{-1/(1-\Psi')}\Psi'', \tag{27}$$

or,[9] using the change of variable,

$$v(x)=\frac{-1}{1-\Psi'}, \qquad dv=\frac{-1}{(1-\Psi')^2}\Psi''\,dx, \tag{28}$$

we have

$$dx=-Kve^v dv. \tag{29}$$

Now (29) can be integrated on the left-hand side from x_0, the lower endpoint of the interval on which we want a solution, up to x, and from $v(x_0)$ to $v(x)$ on the right yielding

$$(x-x_0)=-K(v-1)e^v\big|_{v(x_0)}^{v(x)}, \tag{30}$$

or

$$x-x_0=-K(v(x)-1)e^{v(x)}+K(v(x_0)-1)e^{v(x_0)}. \tag{31}$$

---

[8]See Ames (1968, ch. 2).

[9]This transformation involves dividing by Ψ', and at $\Psi'=0$, this is ill-defined. However, as $\Psi'=0$ for only one value of x, this is not of consequence when the integration is performed to yield (32), below.

To simplify this further, we can use the definition of K in terms of Ψ and Ψ', (28), together with the definition of v, (28), to write K in terms of $\Psi(x)$ and $v(x)$,

$$K = \Psi(x)(1 - \Psi'(x)) e^{1/(1 - \Psi'(x))}, \tag{32}$$

or

$$K = -\Psi(x)\frac{1}{v(x)} e^{-v(x)}. \tag{33}$$

Since (33) is an identity in x we can use it twice in (31), evaluated at x and at x_0,

$$x - x_0 = \Psi(x)\left(\frac{v(x) - 1}{v(x)}\right) - \Psi(x_0)\left(\frac{v(x_0) - 1}{v(x_0)}\right), \tag{34}$$

or, using the definition of v,

$$x - x_0 = \Psi(x)(2 - \Psi'(x)) - \Psi(x_0)(2 - \Psi'(x_0)), \tag{35}$$

or, finally,

$$\Psi'(x) = 2 - \frac{x - \{x_0 - \Psi(x_0)(2 - \Psi'(x_0))\}}{\Psi(x)}. \tag{36}$$

Eq. (36) is a first-order differential equation, linear in Ψ' although non-linear in Ψ and x. Actually, it really describes a family of such equations because $\Psi'(x_0)$ as well as $\Psi(x_0)$ can be specified arbitrarily.

However, although (36) specifies the evolution of $\Psi(\cdot)$ at points x where $\Psi(x) \neq 0$, its non-linear solutions cannot be extended beyond a bounded interval. The extent of this domain of $\Psi(\cdot)$ is determined by the choice of $\Psi(x_0)$ and $\Psi'(x_0)$. Consider (31). Since $K > 0$ and $v(x) < 0$, the first term on the right-hand side is positive. Moreover, since for positively signed solutions $v(x)$ converges to zero (from below) as x increases, the first term is decreasing in x. Thus

$$x - x_0 \leqq K(1 + (v(x_0) - 1)e^{v(x_0)}) \tag{37}$$

serves to define the domain of x. Using (33) (at x_0) to eliminate K we have from (37) that

$$(x - x_0) \leqq \Psi(x_0)(\Psi'(x_0) - 2 + (1 - \Psi'(x_0))e^{1/(1 - \Psi'(x_0))}). \tag{38}$$

The implication of this restriction on the domain of $\Psi(\cdot)$ for the design of contracts is as follows: Given any joint distribution of (ε, δ), we must choose

$\Psi(x_0)$ and $\Psi'(x_0)$ so that $x = \varepsilon - \delta$ satisfies (38) with probability one. This constrains the expression in brackets in (36) and leads to pointwise higher solutions the larger the required domain.

Finally, let us write eq. (36) in the form

$$\Psi'(x) = 2 + \frac{\alpha - x}{\Psi(x)}, \tag{39}$$

in which α is a parameter, and in which the initial conditions are not given explicitly.[10] The solutions to this equation can be parametrized explicitly. Indeed, (39) with $\alpha = 0$ can be written as

$$\frac{\Psi'(x) - 1}{\Psi(x) - x} = \frac{\Psi'(x)}{\Psi(x) - x} - \frac{(\Psi'(x) - 1)\Psi(x)}{(\Psi(x) - x)^2}, \tag{40}$$

which upon integration yields

$$\Psi(x) = -\beta \pm \beta \exp\left\{\frac{\Psi(x)}{\Psi(x) - x}\right\}. \tag{41}$$

This can be parameterized as

$$x(u) = \beta(u + 1)e^{-u}, \qquad y(u) = \Psi(x) = \beta u\, e^{-u} \tag{42}$$
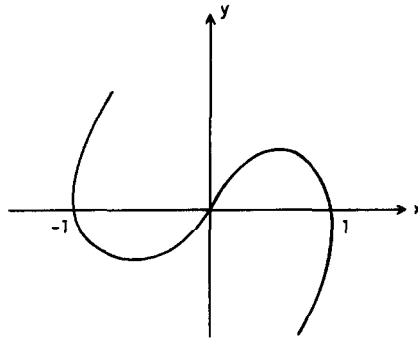

Fig. 2. Graph of (43).

[10]Note that (39) can be derived directly from (17) by writing (17) in the form

$$1 - 2\Psi'(x) + (d/dx)\Psi(x)\Psi'(x) = 0,$$

which then easily gives (39). Our longer derivation has the advantage that the role of the initial conditions is made explicit throughout. The bounds (37) and (38) are utilized in the illustrative numerical computations in section 7.

where β is a constant parameterizing class of solutions.

The solutions are thus simple transformations of

$$x(u) = (u+1)e^{-u}, \qquad y(u) = ue^{-u}. \tag{43}$$

Note that the domain $x \in [0,1]$ is relevant for positive solutions and the domain $x \in [-1,0]$ for negatively signed solutions.

5. Two-signed solutions to (17)

In the previous section we derived a parameterized expression for the solution of our fundamental equation (17). From a technical point of view the form (42) suggests that the class of two-signed solutions can be generated from it 'gluing' together a negative and a positive solution, defined on subsequent subintervals. [Compare also the graph of formula (43), fig. 2.] To see whether this is possible it is necessary to analyze the behavior of a positive solution when one extends it backwards until $\Psi \to 0$. Using the parametric form (42) it is seen that the limit is obtained when $u \to +\infty$, where $\beta > 0$ in the case of a positive solution.

When $u \to +\infty$, we obtain from (42) the limits $x \to \alpha$, and $\Psi \to 0$. The next step is to consider the limiting values for the derivatives $\Psi'(x)$, and $\Psi''(x)$. One easily computes the formulae

$$\Psi'(x) = (dy/du) \div (dx/du) = 1 - u^{-1} \to 1, \tag{44}$$

and

$$\Psi''(x) = -(1 - \Psi')^2/\Psi = e^u u^{-3} \to +\infty. \tag{45}$$

as $u \to +\infty$. For negative solutions these computations hold with β taken to have a negative value.

In total we obtain the conclusion that, if in the parametric form (42) one sets identical values for the α parameter partly specifying the negative and positive solutions from their classes, then one can piece together such solutions to obtain a two-signed solution. This two-signed solution is of class C^1, but somewhat surprisingly it does not possess the second derivative at the point x_0 in which $\psi(x_0) = 0$, unless linear solutions (21) are being glued together.[11] However, elsewhere in the domain the two-signed solution is twice continuously differentiable. Though this phenomenon violates our earlier requirement a little bit, we will consider the two-signed solutions in the next section.

[11]Note that $\Psi(x)\Psi''(x) + 0$, as $u \to +\infty$, which in conjunction with the fact $\Psi'(x) + 1$ suggests that the fundamental equation (17) does not become absurd in the limiting case.

6. Characteristics of the payment function

In the preceding sections we have analyzed the nature of possible solutions to the fundamental equation (17) derived from (10). These results are important in that they permit a qualitative characterization of the cost function $C(q,\delta)$ of the buyer and the revenue function $R(q,\varepsilon)$ of the seller with respect to the traded quantity q.

Given a contract $t = t(\varepsilon,\delta)$, $q = q(\varepsilon,\delta)$, we can, by utilizing the implicit function theorem, solve the latter to obtain $\varepsilon = \varepsilon(q,\delta)$, because $q_\varepsilon > 0$ by the second-order conditions. Substituting this to the former we obtain the cost function of the buyer $C(q,\delta) = t[\varepsilon(q,\delta),\delta]$. By differentiating once

$$\frac{\partial C(q,\delta)}{\partial q} = \frac{t_\varepsilon}{q_\varepsilon},$$

and twice

$$\frac{\partial^2 C(q,\delta)}{\partial q^2} = \frac{t_{\varepsilon\varepsilon}}{(q_\varepsilon)^2} + t_\varepsilon \frac{\partial^2 \varepsilon}{\partial q^2}. \tag{46}$$

To evaluate (46) we have t_ε from (6), $t_{\varepsilon\varepsilon}$ obtained by differentiating (6), and compute

$$\frac{\partial^2 \varepsilon}{\partial q^2} = \frac{\partial}{\partial q}\left[\frac{1}{q_\varepsilon(\varepsilon(q,\delta),\delta)}\right] = -\frac{q_{\varepsilon\varepsilon}}{(q_\varepsilon)^3}. \tag{47}$$

In total we have the result

$$\frac{\partial^2 C(q,\delta)}{\partial q^2} = a + \frac{1}{q_\varepsilon}, \tag{48}$$

the sign of which is in general uncertain. For the special case of solutions dependent on $(\varepsilon - \delta)$ it is possible to rewrite (48) in terms of the function Ψ which was analyzed in the preceding sections. The conclusion is that $t[q,\delta]$ is convex (resp. concave) in q whenever $\Psi' > (\text{resp.} <) - b/a$.

A similar computation for the seller's revenue function yields the result

$$\frac{\partial^2 R(q,\varepsilon)}{\partial q^2} = -b + \frac{1}{q_\delta}, \tag{49}$$

which in turn implies that $t[q;\varepsilon]$ is convex (concave resp.) in q whenever $\Psi' < (\text{resp.} >) - a/b$.

These results can be interpreted easily by noting that, for example, the convexity of $C(q,\delta)$ in q means that the unit price for the buyer is rising, i.e., *quantity premia* appear. In the same vein, whenever $C(q,\delta)$ is concave in q the

contract stipulates *quantity discounts*, i.e., the unit price for the buyer is decreasing.

This analysis can be conveniently related to the different types of solution Ψ of the basic differential equation (17). Consider first a positively signed solution discussed in section 4 above. For then Ψ' is first positive, but as $x(=\varepsilon-\delta)$ increases it turns more and more negative which implies that $R(q,\varepsilon)$ is first convex but it eventually becomes concave. Therefore in these contracts the buyer faces quantity premia at low levels of the traded quantity but discounts appear at high volumes of trading. In section 7 numerical computations illustrate this phenomenon. For negatively signed contracts the conclusion is reversed, i.e., quantity discounts appear at low levels of trading.

As shown in section 5, two-signed contracts are pieced from one-signed contracts on subintervals so that with them quantity discounts are present at both sufficiently low and high levels of trading, while in the intermediate range quantity premia are the rule. These features are illustrated in fig. 3.
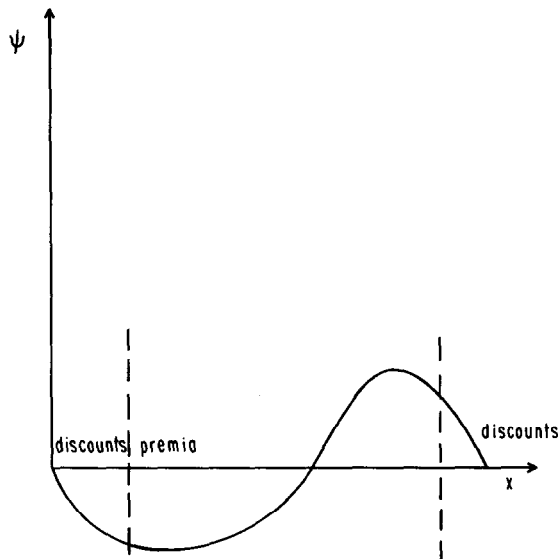

Fig. 3

7. Numerical computations

By virtue of the results in the last three sections a numerical computation of the solutions to (17) can be derived from the one-signed solutions over fixed intervals $(x_0, x_{\max})$ with the properties that

$$\lim_{x \to x_0} \Psi = \lim_{x \to x_{\max}} \Psi = 0.$$

In this section we examine some qualitative properties of these solutions.

Though we are not concerned in obtaining optimal contracts our computations show the interesting feature that it is relatively easy to find a solution to (17) that is 'almost' the pointwise minimizer of Ψ throughout 'almost' all of the domain of definition. Therefore, the calculations at the second stage are carried out for this particular solution.

We now describe the numerical method used and present an illustrative calculation. Let the length of the interval over which we seek a solution be fixed at $L = x_{max} - x_0$. From (38) we have an implicit relationship between $\Psi(x_0)$ and $\Psi'(x_0)$ that must be satisfied if the solution is to be well-defined over this interval,

$$\Psi(x_0) \geqq L/\Psi'(x_0) - 2 + (1 - \Psi'(x_0))e^{1/(1-\Psi'(x_0))}. \qquad (50)$$

Thus for fixed L we consider the one parameter family of solutions to (36) where $\Psi'(x_0)$ is fixed arbitrarily in $(0, 1)$ and $\Psi(x_0)$ is given by the solution to (50) with equality. A standard computer program for numerical integration was used to integrate the expression for $\Psi'(x)$ from x_0 to x.

Fig. 4 displays the results for various choices of $\Psi'(x_0)$ when $L = 10$. The central feature of this simulation is that the solution obtained for $\Psi'(x_0) = 0.9$
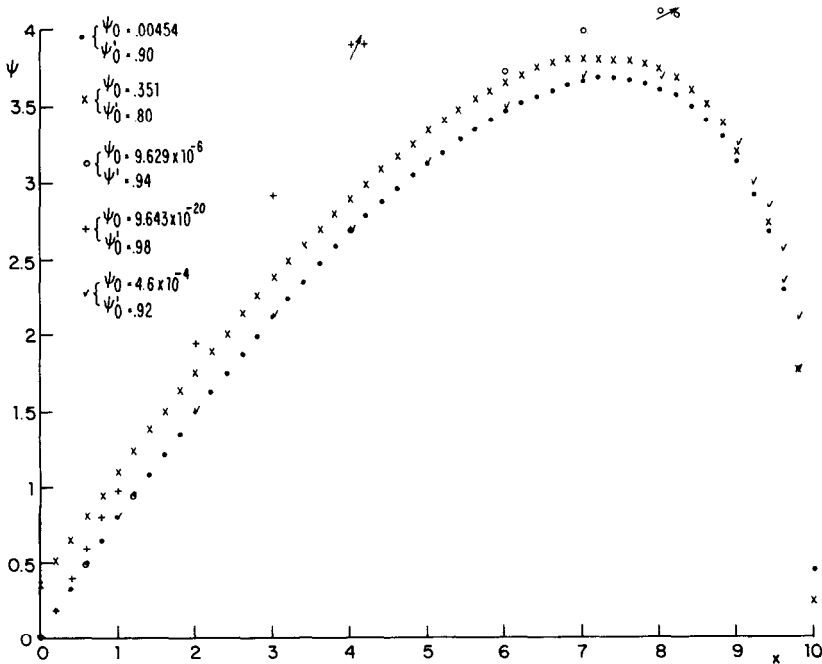

Fig. 4

is 'almost' the pointwise minimizer of the family of all solutions obtained for
values of $\Psi'(x_0)$ in steps of 0.02. Only for $x - x_0 \leqq 0.2$ or $x - x_0 \geqq 9.8$ were any
of the other solutions below this one.

This result seems robust to the length of the interval and to step sizes of
$\Psi'(x_0)$ used to construct the families of solutions. It is probably a good
approximation to the 'best' one-signed solution over this interval for most
distributions of x.

We then computed the non-linear price and revenue functions that are
implicit in the optimal contract, using this result as an approximation for the
optimum. This computation was compatible with the results of section 6 on
the concavity properties of these functions.

The numerical $\Psi(\cdot)$ obtained above was substituted into the expressions

$$t_\varepsilon(\varepsilon, \delta) = \frac{(a\Psi + a\delta + b\varepsilon)(c_0 + \Psi - (\varepsilon - \delta))}{\Psi(a + b)^2}, \tag{51}$$

$$t_\delta(\varepsilon, \delta) = \frac{(b\Psi - b\varepsilon - a\delta)(c_0 + \Psi - (\varepsilon - \delta))}{\Psi(a + b)^2},$$

where $c_0 = x_0 - \Psi(x_0)(2 - \Psi'(x_0))$. Both a and b were set at -1. These
partial derivatives were integrated numerically over the rectangle
$(\varepsilon, \delta) \in ([5, 10] \times [0, 5])$ so that, as required, $\varepsilon - \delta \in [0, 10]$. The value of $t(5, 0)$
$= 0$ was taken as a normalization without loss of generality.

From this function $t(\varepsilon, \delta)$ we computed the price and revenue functions as
follows. For each value of δ and each $q \in ((5 - \delta)/2, (10 - \delta)/2)$ we find the
value of ε so that $-1/2(\Psi(\varepsilon - \delta) - \varepsilon - \delta) = q$. This gives us a function $\varepsilon(q, \delta)$
whose interpretation is that it is the 'announcement' of ε which, when
combined with the given value of δ, would induce the given value of q to be
exchanged under the contract Ψ. Finally, the total cost to the buyer of
quantity $C(q, \delta)$, is defined by $C(q, \delta) = t(\varepsilon(q, \delta), \delta)$. The revenue functions of the
seller facing a buyer whose announced parameter is ε, $R(q, \varepsilon)$ is given
symmetrically.

Some of these cost and revenue functions are given in figs. 5 and 6. It is
noteworthy that the quantity discount/quantity premia results of section 6
are verified in this numerical construction.

Because we cannot find the true optimum without knowing the
distribution of x, and, more importantly, because the incentive compatible
contracts we have studied here are constrained to divide the surplus evenly
between the players in all circumstances, these numerical results should be
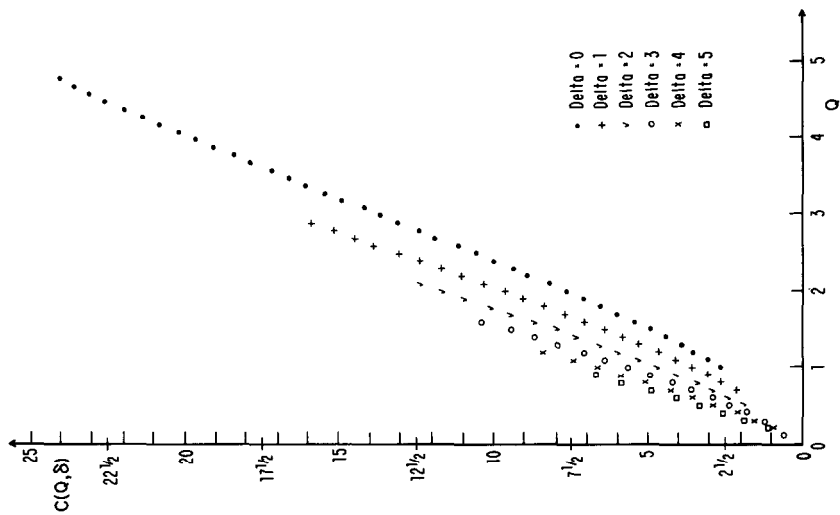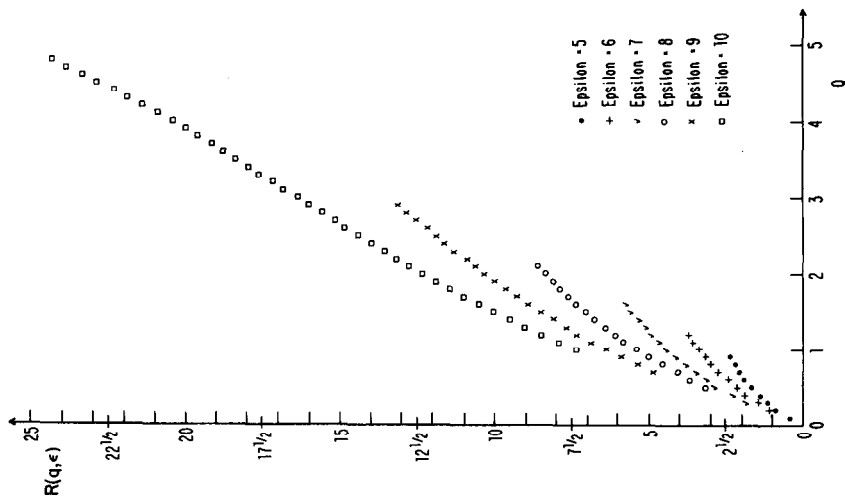viewed as merely illustrative.

Fig. 6

Fig. 5

References

Ames, W.F., 1968, Non-linear ordinary differential equations in transport processes (Academic Press, New York).

Green, J. and J.-J. Laffont, 1979, Incentives in public decision making (North-Holland, Amsterdam).

Hellwig, M., 1960, Partial differential equations (Blaisdell, New York).

Laffont, J.-J. and E. Maskin, 1980, A differential approach to dominant strategy mechanisms, Econometrica 48, 1507–1520.

Spence, A.M., 1977, Non-linear prices and welfare, Journal of Public Economics 8, Aug., 1–18.

Weitzman, M., 1974, Prices vs. quantities, Review of Economic Studies, 477–492.